Computer Aided Geometric Design

Differential Geometry of Surfaces

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based on a book by Cohen, Riesenfeld, & Elber

Regular Surfaces

Question: What is the condition for a curve to be regular?

Definition 12.1:

A regular parametric representation of class $C^{(k)}$, for $k \geq 1$, of a set of points W in R^3 is a mapping f: $U \rightarrow W$, where U is an open set in R^2 and f is on W, and:

$$\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \neq 0.$$

Example 12.2

For $f : R^2 \rightarrow R$, a scalar valued bivariate function, let

$\boldsymbol{f}(\boldsymbol{x}) = (x_1, x_2, f(\boldsymbol{x}))$, $\boldsymbol{x} = (x_1, x_2)$. If f is $C^{(k)}$, so is $\boldsymbol{f}$.

Note that $\dfrac{\partial \boldsymbol{f}}{\partial x_1} = \left(1, 0, \dfrac{\partial f}{\partial x_1}\right)$ and $\dfrac{\partial \boldsymbol{f}}{\partial x_2} = \left(0, 1, \dfrac{\partial f}{\partial x_2}\right)$.

Then, $\dfrac{\partial \boldsymbol{f}}{\partial x_1} \times \dfrac{\partial \boldsymbol{f}}{\partial x_2} = \left(-\dfrac{\partial f}{\partial x_1}, -\dfrac{\partial f}{\partial x_2}, 1\right)$, which can never result in the zero vector. Hence, an explicit $C^{(1)}$ surface can always be represented as a regular parametric representation.

Example 12.4

Consider $\boldsymbol{U} = \{ \boldsymbol{u} : \| \boldsymbol{u} \| < 1 \}$ with

$$\boldsymbol{f} = \left(u_1, u_2, \sqrt{1 - (u_1)^2 - (u_2)^2} \right).$$

This is a regular parametric representation. Why?

Question: What is this surface?

Theorem 12.5

Let $f : R^2 \to R$. For $\boldsymbol{a} = (a_1, a_2)$ in the domain of f, define $C_{\boldsymbol{a},r} = \{ \boldsymbol{x} : \| \boldsymbol{x} - \boldsymbol{a} \| < r \}$. Assume f is $C^{(n+1)}$, on $C_{\boldsymbol{a},r}$. Then, for $\boldsymbol{x} \in C_{\boldsymbol{a},r}$ and $\delta_{\boldsymbol{x}} = (\delta_1, \delta_2) = (x_1 - a_1, x_2 - a_2)$,

$$f(\boldsymbol{x}) = \sum_{j=0}^{n} \frac{1}{j!} \left(\delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} \right)^j f(\boldsymbol{a}) + R_{\boldsymbol{a},n},$$

where

$$\left(\delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} \right)^j = \sum_{i=0}^{j} \binom{j}{i} (\delta_1)^i (\delta_2)^{j-i} \frac{\partial^j}{\partial x_1{}^i \partial x_2{}^{j-i}}.$$

Theorem 12.5 (Cont.)

and,

$$R_{\boldsymbol{a},n} = \int_0^1 \frac{(1-t)^n}{n!} \left(\delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} \right)^{n+1} f(a_1 + \delta_1 t, a_2 + \delta_2 t) dt,$$

$$= \frac{1}{(n+1)!} \left(\delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} \right)^{n+1} f(a_1 + \delta_1 \theta, a_2 + \delta_2 \theta),$$

where θ is a certain value between zero and one.

$$f(\boldsymbol{u}(\boldsymbol{v})) = f(u_1(v_1, v_2), u_2(v_1, v_2))$$

Allowable Change of Parameter

Question: What are the condition for $u(t)$ in $C(u(t))$ to be an allowable change of parameter of regular curve $C(u)$?

Consider the surface $f(\boldsymbol{u}(\boldsymbol{v}))$, where f is a regular surface,

$$\frac{\partial f}{\partial v_1} \times \frac{\partial f}{\partial v_2} = \left(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial v_1} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial v_1} \right) \times \left(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial v_2} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial v_2} \right)$$

$$= \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial v_1}\frac{\partial u_1}{\partial v_2} + \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}\frac{\partial u_1}{\partial v_1}\frac{\partial u_2}{\partial v_2}$$

$$+ \frac{\partial f}{\partial u_2} \times \frac{\partial f}{\partial u_1}\frac{\partial u_2}{\partial v_1}\frac{\partial u_1}{\partial v_2} + \frac{\partial f}{\partial u_2} \times \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial v_1}\frac{\partial u_2}{\partial v_2}$$

Allowable Change of Parameter

$$\frac{\partial f}{\partial v_1}\times\frac{\partial f}{\partial v_2}=\frac{\partial f}{\partial u_1}\times\frac{\partial f}{\partial u_2}\frac{\partial u_1}{\partial v_1}\frac{\partial u_2}{\partial v_2}+\frac{\partial f}{\partial u_2}\times\frac{\partial f}{\partial u_1}\frac{\partial u_2}{\partial v_1}\frac{\partial u_1}{\partial v_2}$$

$$=\frac{\partial f}{\partial u_1}\times\frac{\partial f}{\partial u_2}\frac{\partial u_1}{\partial v_1}\frac{\partial u_2}{\partial v_2}-\frac{\partial f}{\partial u_1}\times\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial v_1}\frac{\partial u_1}{\partial v_2}$$

$$=\frac{\partial f}{\partial u_1}\times\frac{\partial f}{\partial u_2}\left(\frac{\partial u_1}{\partial v_1}\frac{\partial u_2}{\partial v_2}-\frac{\partial u_2}{\partial v_1}\frac{\partial u_1}{\partial v_2}\right)$$

$$=\frac{\partial f}{\partial u_1}\times\frac{\partial f}{\partial u_2}\frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}.$$

Hence, we must require for the Jacobian to be non zero.

Definition 12.9

A $C^{(k)}$ simple surface, also known as a coordinate patch, is a regular parametric representation that is one-to-one function.

Example 12.10

Consider planar regular curve $(x_1(t), 0, x_3(t))$ with $x_1(t) > 0$.

Define $\boldsymbol{f}(t,\theta) = (x_1(t)\cos(\theta), x_1(t)\sin(\theta), x_3(t))$ where $(t,\theta) \in R \times R_b$.

The surface $\boldsymbol{f}$ is called a surface of revolution.

$$\frac{\partial \boldsymbol{f}}{\partial t} = \left(\frac{\partial x_1}{\partial t}\cos\theta, \frac{\partial x_1}{\partial t}\sin\theta, \frac{\partial x_3}{\partial t}\right)$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta} = \left(-x_1\sin\theta, x_1\cos\theta, 0\right)$$

$$\frac{\partial \mathbf{f}}{\partial t} = \left(\frac{\partial x_1}{\partial t}\cos\theta, \frac{\partial x_1}{\partial t}\sin\theta, \frac{\partial x_3}{\partial t} \right)$$

Example 12.10 (Cont.)

$$\frac{\partial \mathbf{f}}{\partial \theta} = \left(-x_1 \sin\theta, x_1 \cos\theta, 0 \right)$$

or $$\frac{\partial \mathbf{f}}{\partial t} \times \frac{\partial \mathbf{f}}{\partial \theta} = \left(-x_1 \frac{\partial x_3}{\partial t}\cos\theta, -x_1 \frac{\partial x_3}{\partial t}\sin\theta, x_1 \frac{\partial x_1}{\partial t}\cos^2\theta + x_1 \frac{\partial x_1}{\partial t}\sin^2\theta \right)$$

$$= \left(-x_1 \frac{\partial x_3}{\partial t}\cos\theta, -x_1 \frac{\partial x_3}{\partial t}\sin\theta, x_1 \frac{\partial x_1}{\partial t} \right),$$

and $$\left\| \frac{\partial \mathbf{f}}{\partial t} \times \frac{\partial \mathbf{f}}{\partial \theta} \right\| = \sqrt{(x_1)^2 \left(\frac{\partial x_3}{\partial t} \right)^2 \cos^2\theta + (x_1)^2 \left(\frac{\partial x_3}{\partial t} \right)^2 \sin^2\theta + (x_1)^2 \left(\frac{\partial x_1}{\partial t} \right)^2}$$

$$= x_1 \sqrt{\left(\frac{\partial x_3}{\partial t} \right)^2 + \left(\frac{\partial x_1}{\partial t} \right)^2}.$$

Question: What is the angle between $\frac{\partial \mathbf{f}}{\partial t}$ and $\frac{\partial \mathbf{f}}{\partial \theta}$?

$$\frac{\partial f}{\partial t} = \left(\frac{\partial x_1}{\partial t}\cos\theta, \frac{\partial x_1}{\partial t}\sin\theta, \frac{\partial x_3}{\partial t} \right)$$

$$\frac{\partial f}{\partial \theta} = \left(-x_1 \sin\theta, x_1 \cos\theta, 0 \right)$$

Example 12.10 (Cont.)

But $(x_1(t), 0, x_3(t))$ is a regular curve with $x_1(t) > 0$. Hence,

$$\left\| \frac{\partial f}{\partial t} \times \frac{\partial f}{\partial \theta} \right\| = x_1 \sqrt{\left(\frac{\partial x_3}{\partial t} \right)^2 + \left(\frac{\partial x_1}{\partial t} \right)^2} > 0.$$

and f is a simple surface. Furthermore,

$$\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial \theta} \right\rangle = \left(-x_1 \frac{\partial x_1}{\partial t}\sin\theta\cos\theta + x_1 \frac{\partial x_1}{\partial t}\sin\theta\cos\theta \right) = 0,$$

or the partials are orthogonal.

Lemma 12.19

If f is a coordinate patch, then

$$\left\{ \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\}$$

forms a basis for R^3.

Proof: Follows immediately from the independence of these three vectors.

Tangent to Surfaces (Section 12.2)

Consider $\gamma(t) = \boldsymbol{f}(u_1(t), u_2(t))$ where $\boldsymbol{f}$ is a regular surface and $(u_1(t), u_2(t))$ is a regular planar curve. By the chain rule,

$$\frac{\partial \boldsymbol{f}}{\partial t} = \frac{\partial \boldsymbol{f}}{\partial u_1}\frac{\partial u_1}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial u_2}\frac{\partial u_2}{\partial t}.$$

Because $\boldsymbol{f}$ is a regular surface, $\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \neq 0$, so $\frac{\partial f}{\partial u_1} \neq 0$ and $\frac{\partial f}{\partial u_2} \neq 0$.

Further, since $(u_1(t), u_2(t))$ is a regular curve, $\left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right) \neq (0,0)$.

Thus, $d\boldsymbol{f}/dt \neq 0$ for all t values and $\gamma(t)$ is a regular curve.

Definition 12.21

The vector space of the tangent plane $T_{f,a}$ to a simple surface $f : U \to R^3$ at a point $f(a)$ is the plane spanned by $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ at a. Thus, this plane is the plane through the origin with normal vector $\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}$.

The tangent plane of f at $f(a)$ is the plane through the point $f(a)$ with normal vector $\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}$.

Directional Derivative

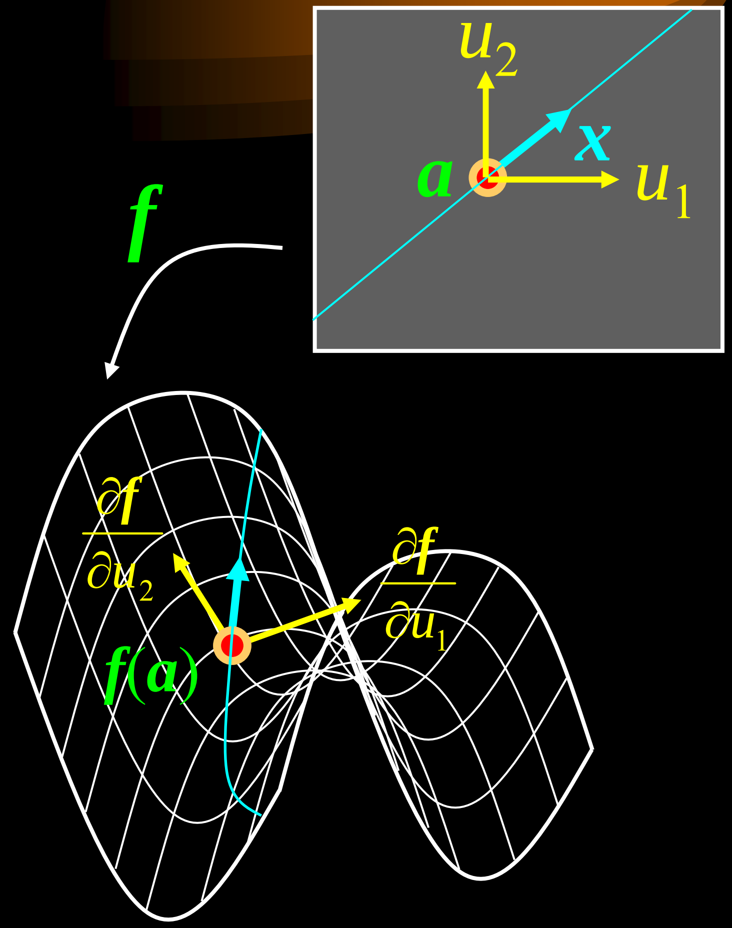
For a given direction $\boldsymbol{x} = (x_1, x_2)$, consider $(u_1(t), u_2(t)) = (a_1 + t\, x_1, a_2 + t\, x_2)$. We already know that this is a regular curve, and hence, for a simple surface $\boldsymbol{f}$, $\boldsymbol{f}(u_1(t), u_2(t))$ is a regular curve with derivative

$$\frac{\partial \boldsymbol{f}}{\partial t} = x_1 \frac{\partial \boldsymbol{f}}{\partial u_1} + x_2 \frac{\partial \boldsymbol{f}}{\partial u_2}.$$

Lemma 12.22

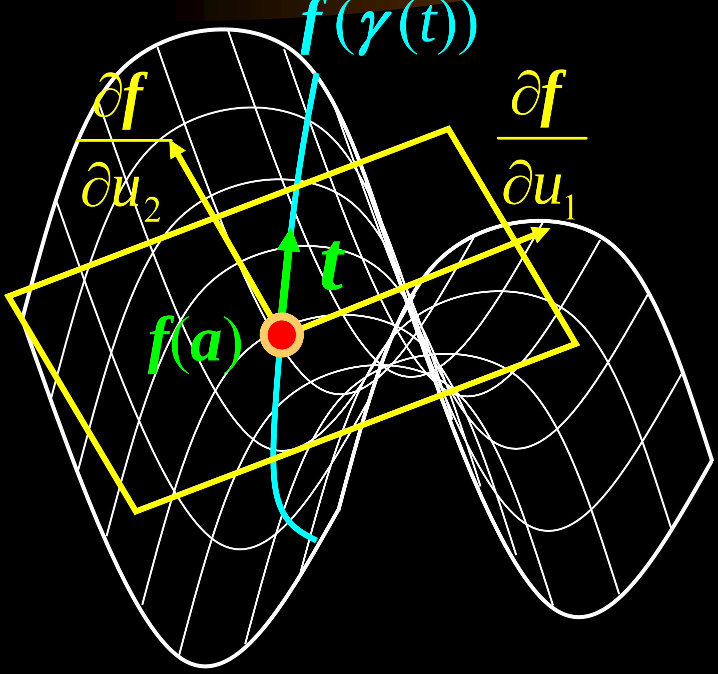
The directional derivative of a simple surface $\boldsymbol{f}$ in the direction $\boldsymbol{x} = (x_1, x_2)$, at the point $\boldsymbol{a} = (a_1, a_2)$, is,

$$D_x \boldsymbol{f}(\boldsymbol{a}) = x_1 \frac{\partial \boldsymbol{f}}{\partial u_1}(\boldsymbol{a}) + x_2 \frac{\partial \boldsymbol{f}}{\partial u_2}(\boldsymbol{a}).$$



Lemma 12.24

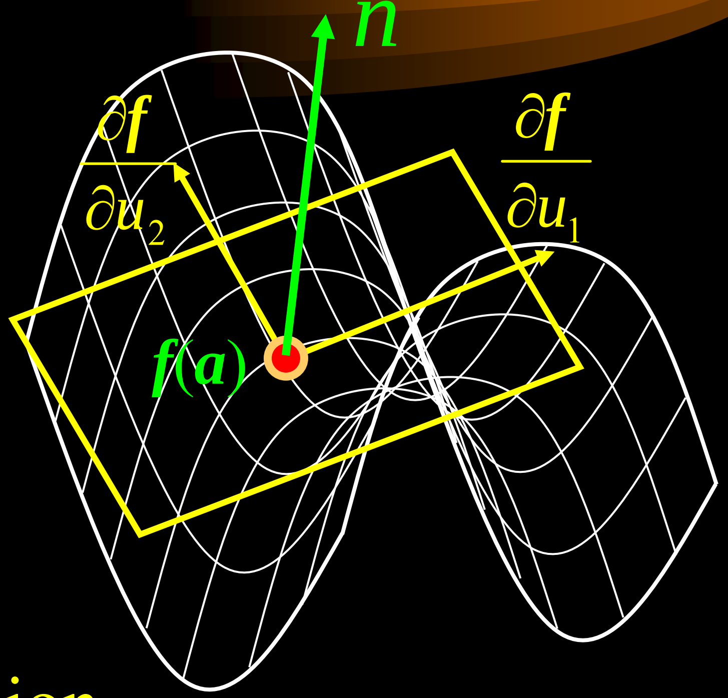
A vector $\boldsymbol{t}$ is in the vector space of the tangent plane of the simple surface $\boldsymbol{f}$ at $\boldsymbol{f}(\boldsymbol{a})$ if and only if it is a tangent vector of some regular curve $\boldsymbol{f}(\gamma(t))$ in surface $\boldsymbol{f}$ at the point $\boldsymbol{f}(\boldsymbol{a})$.



Definition 12.25

The unit normal to the regular surface $\boldsymbol{f}$ at $\boldsymbol{f}(\boldsymbol{a})$ is

$$n = \frac{\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2}}{\left\| \frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} \right\|}.$$



Corollary 12.26: the dimension of the vector space of the tangent plane is two.

First Fundamental Form (Section 12.3)

Consider $\gamma(t) = f(u_1(t), u_2(t))$, an arbitrary regular curve in the simple surface f. Recall that the arc length between $t = a$ and $t = t$ equals $\int_a^t \left\| \frac{d\gamma(\tau)}{d\tau} \right\| d\tau$.

We also know that the rate of change of the arc length is $\left\| \frac{d\gamma(t)}{dt} \right\|$.

$$\gamma(t) = \boldsymbol{f}(u_1(t), u_2(t))$$

First Fundamental Form (Cont.)

Hence
$$\left(\frac{ds}{dt}\right)^2 = \left\|\frac{d\gamma(t)}{dt}\right\|^2 = \left\langle \frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \right\rangle$$
$$= \left\langle \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_i}{dt}, \sum_j \frac{\partial \boldsymbol{f}}{\partial u_j}\frac{du_j}{dt} \right\rangle$$
$$= \sum_i \sum_j \frac{du_i}{dt} \left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \frac{\partial \boldsymbol{f}}{\partial u_j} \right\rangle \frac{du_j}{dt}$$
$$= \sum_i \sum_j \frac{du_i}{dt} g_{ij} \frac{du_j}{dt} = \left[\frac{du_1}{dt}, \frac{du_2}{dt}\right] \boldsymbol{G} \left[\frac{du_1}{dt}, \frac{du_2}{dt}\right]^T$$

First Fundamental Form (Cont.)

where $g_{ij} = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$ and $\boldsymbol{G} = (g_{ij})$.

By symmetry properties of inner products, $g_{12} = g_{21}$.

The above quadratic form can be expanded to,

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du_1}{dt}\right)^2 + 2F\frac{du_1}{dt}\frac{du_2}{dt} + G\left(\frac{du_2}{dt}\right)^2$$

$$= \frac{1}{E}\left[\left(E\frac{du_1}{dt} + F\frac{du_2}{dt}\right)^2 + \left(EG - F^2\right)\left(\frac{du_2}{dt}\right)^2\right],$$

where $E = g_{11}$, $F = g_{12}$, and $G = g_{22}$.

First Fundamental Form (Cont.)

The quantity of $EG - F^2$ is just $\det(\mathbf{G}) = |\mathbf{G}|$.

Question: When is a quadratic form called positive definite?

Because the arc length is a strictly increasing function for all regular curves, this quadratic form must be positive. $E > 0$ and $G > 0$ (why?) and further $EG - F^2 = \det(\mathbf{G}) = |\mathbf{G}| > 0$.

Definition 12.28

The quantity

$$E\left(\frac{du_1}{dt}\right)^2 + 2F\frac{du_1}{dt}\frac{du_2}{dt} + G\left(\frac{du_2}{dt}\right)^2$$

is called the first fundamental form and is frequently denoted by $\mathrm{I}\left(\frac{du_1}{dt}, \frac{du_2}{dt}\right)$.

$$\boldsymbol{f} = \left(u_1, u_2, \sqrt{1-(u_1)^2-(u_2)^2}\right).$$

Example 12.29

Using the simple surface of Example 12.4

$$\frac{\partial \boldsymbol{f}}{\partial u_1}(\boldsymbol{u}) = \left(1, 0, \frac{-u_1}{\sqrt{1-(u_1)^2-(u_2)^2}}\right) \text{ and } \frac{\partial \boldsymbol{f}}{\partial u_2}(\boldsymbol{u}) = \left(0, 1, \frac{-u_2}{\sqrt{1-(u_1)^2-(u_2)^2}}\right).$$

or

$$g_{11} = 1 + \frac{(u_1)^2}{1-(u_1)^2-(u_2)^2} = \frac{1-(u_2)^2}{1-(u_1)^2-(u_2)^2}$$

$$g_{12} = g_{21} = \frac{u_1 u_2}{1-(u_1)^2-(u_2)^2}$$

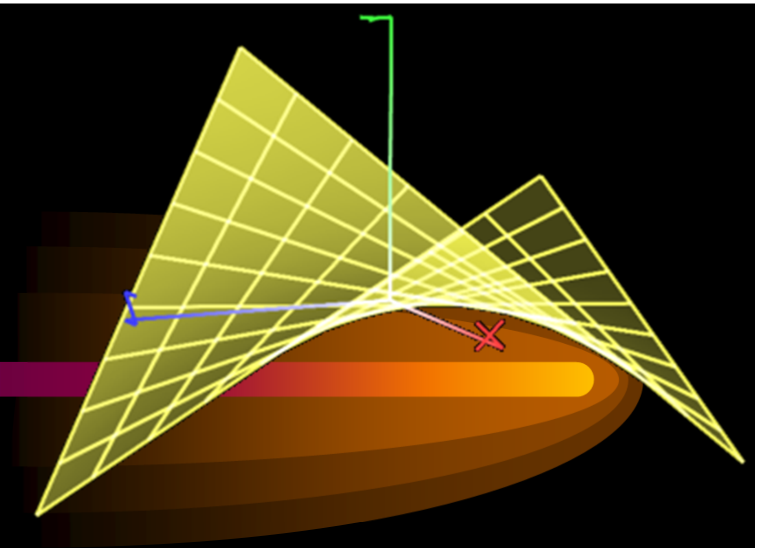
$$g_{22} = 1 + \frac{(u_2)^2}{1-(u_1)^2-(u_2)^2} = \frac{1-(u_1)^2}{1-(u_1)^2-(u_2)^2},$$

Example 12.29 (Cont.)

and

$$\mathbf{G} = \frac{1}{1-(u_1)^2-(u_2)^2}\begin{bmatrix} 1-(u_2)^2 & u_1u_2 \\ u_1u_2 & 1-(u_1)^2 \end{bmatrix}.$$

Example 12.30



For $\boldsymbol{f}(\boldsymbol{u}) = (u_1 + u_2, u_1 u_2, u_1 - u_2)$,

$$\frac{\partial \boldsymbol{f}}{\partial u_1} = (1, u_2, 1) \quad \text{and} \quad \frac{\partial \boldsymbol{f}}{\partial u_2} = (1, u_1, -1).$$

$$\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} = (-u_1 - u_2, 2, u_1 - u_2),$$

which can never be zero. Then,

$$g_{11} = 2 + (u_2)^2 \qquad g_{12} = u_1 u_2 \qquad g_{22} = 2 + (u_1)^2.$$

Invariance of the First Fundamental Form

Suppose that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v})$ is a coordinate transformation and that $\boldsymbol{f}(\boldsymbol{u})$ is a simple surface. $\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{u}(\boldsymbol{v}))$ is another regular parametric representation of the same surface. Let $\boldsymbol{h}(\boldsymbol{v}(t))$ be the curve $\boldsymbol{f}(\boldsymbol{u}(t))$ in the $\boldsymbol{v}$ parameterization. Then,

$$\frac{d\boldsymbol{h}}{dt} = \sum_{i=1}^{2} \frac{\partial \boldsymbol{h}}{\partial v_i} \frac{dv_i}{dt} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial \boldsymbol{f}}{\partial u_j} \frac{du_j}{dv_i} \frac{dv_i}{dt}.$$

Invariance of the First Fundamental Form (Cont.)

Consider the elements of $\boldsymbol{G}^*$, the matrix of the first fundamental form for $\boldsymbol{h}$,

$$g_{ij}^* = \left\langle \frac{\partial \boldsymbol{h}}{\partial v_i}, \frac{\partial \boldsymbol{h}}{\partial v_j} \right\rangle = \left\langle \sum_{k=1}^{2} \frac{\partial \boldsymbol{f}}{\partial u_k} \frac{\partial u_k}{\partial v_i}, \sum_{r=1}^{2} \frac{\partial \boldsymbol{f}}{\partial u_r} \frac{\partial u_r}{\partial v_j} \right\rangle$$

$$= \sum_{k=1}^{2} \sum_{r=1}^{2} \left\langle \frac{\partial \boldsymbol{f}}{\partial u_k} \frac{\partial u_k}{\partial v_i}, \frac{\partial \boldsymbol{f}}{\partial u_r} \frac{\partial u_r}{\partial v_j} \right\rangle = \sum_{k=1}^{2} \sum_{r=1}^{2} \left\langle \frac{\partial \boldsymbol{f}}{\partial u_k}, \frac{\partial \boldsymbol{f}}{\partial u_r} \right\rangle \frac{\partial u_k}{\partial v_i} \frac{\partial u_r}{\partial v_j}$$

$$= \sum_{k=1}^{2} \sum_{r=1}^{2} g_{kr} \frac{\partial u_k}{\partial v_i} \frac{\partial u_r}{\partial v_j} = \left[\frac{\partial u_1}{\partial v_i}, \frac{\partial u_2}{\partial v_i} \right] \boldsymbol{G} \left[\frac{\partial u_1}{\partial v_j}, \frac{\partial u_2}{\partial v_j} \right]^T$$

Invariance of the First Fundamental Form (Cont.)

In general, $\boldsymbol{G}^* = J_v(\boldsymbol{u})\boldsymbol{G}\left(J_v(\boldsymbol{u})\right)^T$

where,

$$J_v(\boldsymbol{u}) = \begin{bmatrix} \frac{\partial u_1}{\partial v_1} & \frac{\partial u_2}{\partial v_1} \\ \frac{\partial u_1}{\partial v_2} & \frac{\partial u_2}{\partial v_2} \end{bmatrix}.$$

Thus, in general, the coefficients of the first fundamental form are not invariant under coordinate transformation.

Invariance of the First Fundamental Form (Cont.)

Now consider $(ds_h/dt)^2$,

$$\left(\frac{ds_h}{dt}\right)^2 = \left\langle \frac{\partial \boldsymbol{h}(\boldsymbol{v}(t))}{\partial t}, \frac{\partial \boldsymbol{h}(\boldsymbol{v}(t))}{\partial t} \right\rangle$$

$$= \left[\frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}\right] \boldsymbol{G}^* \left[\frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}\right]^T$$

$$= \left[\frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}\right] J_v(\boldsymbol{u}) \boldsymbol{G} J_v(\boldsymbol{u})^T \left[\frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}\right]^T.$$

Invariance of the First Fundamental Form (Cont.)

$$J_v(\boldsymbol{u}) = \begin{bmatrix} \frac{\partial u_1}{\partial v_1} & \frac{\partial u_2}{\partial v_1} \\ \frac{\partial u_1}{\partial v_2} & \frac{\partial u_2}{\partial v_2} \end{bmatrix}.$$

But $$\left[\frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}\right] J_v(\boldsymbol{u}) = \left[\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right],$$

and so, $$\left(\frac{ds_h}{dt}\right)^2 = \left[\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right] \boldsymbol{G} \left[\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right]^T.$$

Theorem 12.31: The first fundamental form is invariant under coordinate transformation.

Angles Between Tangent Vectors

Consider the two tangent vectors $\frac{\partial \boldsymbol{f}}{\partial u_1}$ and $\frac{\partial \boldsymbol{f}}{\partial u_2}$.

Question: Are these two tangent vectors orthogonal?

If $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$ are two tangent vectors in $\boldsymbol{T_f}$ at $\boldsymbol{u}$, then one can write,

$$\boldsymbol{t}^1 = t_1^1 \frac{\partial \boldsymbol{f}}{\partial u_1} + t_2^1 \frac{\partial \boldsymbol{f}}{\partial u_2},$$

$$\boldsymbol{t}^2 = t_1^2 \frac{\partial \boldsymbol{f}}{\partial u_1} + t_2^2 \frac{\partial \boldsymbol{f}}{\partial u_2}.$$

Angles Between Tangent Vectors (Cont.)

In order to measure the angle between $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$, consider their inner product,

$$\left\langle \boldsymbol{t}^1, \boldsymbol{t}^2 \right\rangle = \sum_i \sum_j t_i^1 t_j^2 \left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \frac{\partial \boldsymbol{f}}{\partial u_j} \right\rangle$$

$$= \sum_i \sum_j t_i^1 t_j^2 g_{ij} = \left[t_1^2, t_2^2 \right] \boldsymbol{G} \left[t_1^1, t_2^1 \right]^T.$$

Angles Between Tangent Vectors (Cont.)

Let the angle between $\boldsymbol{t}^1$ and $\boldsymbol{t}^2$ be θ. Then,

$$\cos\theta = \frac{\langle \boldsymbol{t}^1, \boldsymbol{t}^2 \rangle}{\|\boldsymbol{t}^1\| \|\boldsymbol{t}^2\|} = \frac{\sum_i \sum_j t_i^1 t_j^2 g_{ij}}{\sqrt{\sum_i \sum_j t_i^1 t_j^1 g_{ij}} \sqrt{\sum_i \sum_j t_i^2 t_j^2 g_{ij}}}.$$

Lemma 12.32

The unnormalized vector $\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2}$ to the simple surface $\boldsymbol{f}$ at $\boldsymbol{u}$ has magnitude

$$\left\| \frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} \right\| = \sqrt{|\boldsymbol{G}|}.$$

Lemma 12.33

The first partial derivatives of surface $\boldsymbol{f}$ at $\boldsymbol{u}$ are orthogonal if and only if $g_{12} = 0$. The i'th partial has unit length if and only if $g_{ii} = 1$.

Proof:

$$\left\langle \frac{\partial \boldsymbol{f}}{\partial u_1}, \frac{\partial \boldsymbol{f}}{\partial u_2} \right\rangle = \left\| \frac{\partial \boldsymbol{f}}{\partial u_1} \right\| \left\| \frac{\partial \boldsymbol{f}}{\partial u_2} \right\| \cos\theta.$$

where θ is the angle between the partials.

Surface Area

Consider surface $\boldsymbol{f}$ at $\boldsymbol{f}(\boldsymbol{a})$ and let du_1 and du_2 be small positive real numbers. Assume $\boldsymbol{f} \in C^{(1)}$. For small enough du_1 and du_2,

$$\frac{\partial \boldsymbol{f}}{\partial u_1} du_1 \approx \boldsymbol{f}(u_1 + du_1, u_2) - \boldsymbol{f}(u_1, u_2)$$

and

$$\frac{\partial \boldsymbol{f}}{\partial u_2} du_2 \approx \boldsymbol{f}(u_1, u_2 + du_2) - \boldsymbol{f}(u_1, u_2).$$

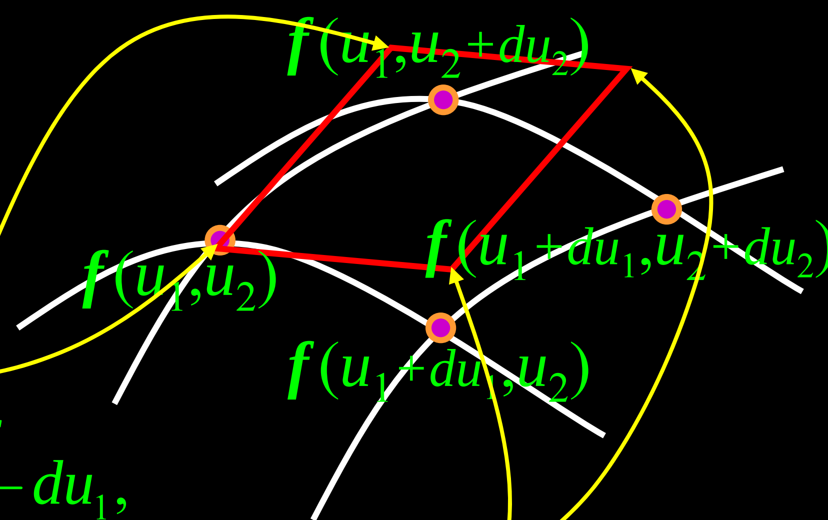
Surface Area (Cont.)

Now consider a surface area element bounded by

$[u_1\ ,\ u_1+du_1]\times[u_2\ ,\ u_2+du_2]$.

These four surface points

can be approximated using,



$$f(u_1,u_2),\qquad f(u_1,u_2)+\frac{\partial f}{\partial u_1}du_1,$$

$$f(u_1,u_2)+\frac{\partial f}{\partial u_2}du_2,\qquad f(u_1,u_2)+\frac{\partial f}{\partial u_1}du_1+\frac{\partial f}{\partial u_2}du_2.$$

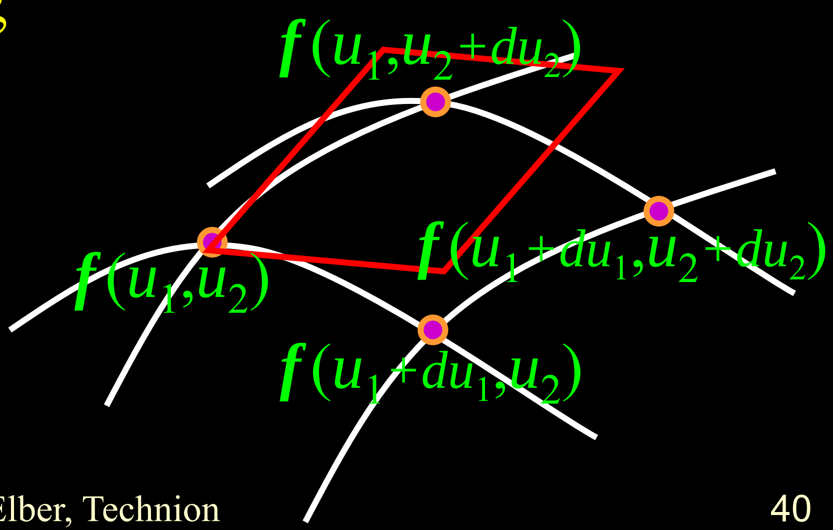
Surface Area (Cont.)

The area of this parallelogram is just

$$\left\|\frac{\partial \boldsymbol{f}}{\partial u_1}du_1 \times \frac{\partial \boldsymbol{f}}{\partial u_2}du_2\right\| = \left\|\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2}\right\| du_1 du_2 = \sqrt{|\boldsymbol{G}|}\, du_1 du_2 .$$

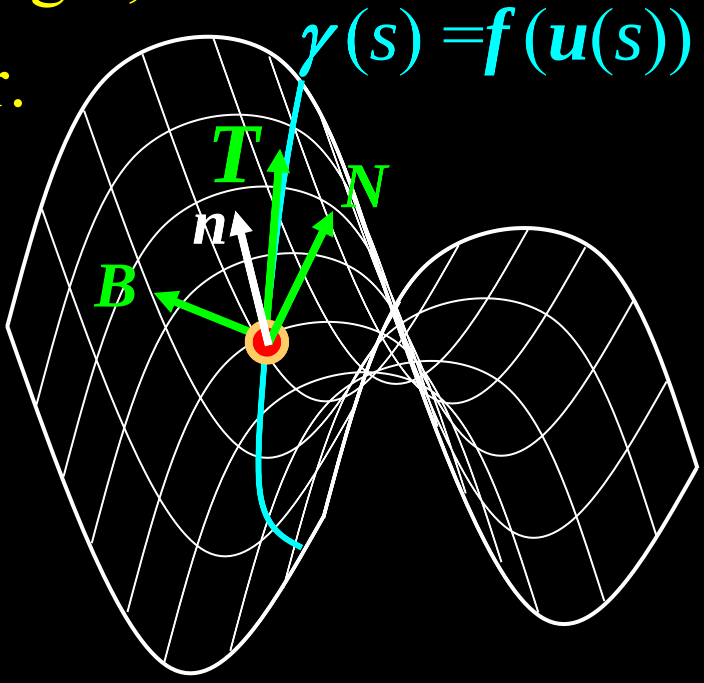
or by integrating and taking the limit,

$$Area = \int\int \sqrt{|\boldsymbol{G}|}\, du_1 du_2 .$$



Second Fundamental Form (Section 12.4)

We will now consider the second order geometry of all curves in the surface. Consider $\gamma(s) = f(u_1(s), u_2(s))$, γ is the assumed parameterized arc length, with s as the arc length parameter. Denote by $\boldsymbol{T}$, the unit tangent vector of $\gamma(s)$, by $\boldsymbol{N}$ and $\boldsymbol{B}$, the unit normal and binormal vector of $\gamma(s)$, and by κ and τ, the curvature and torsion of $\gamma(s)$.



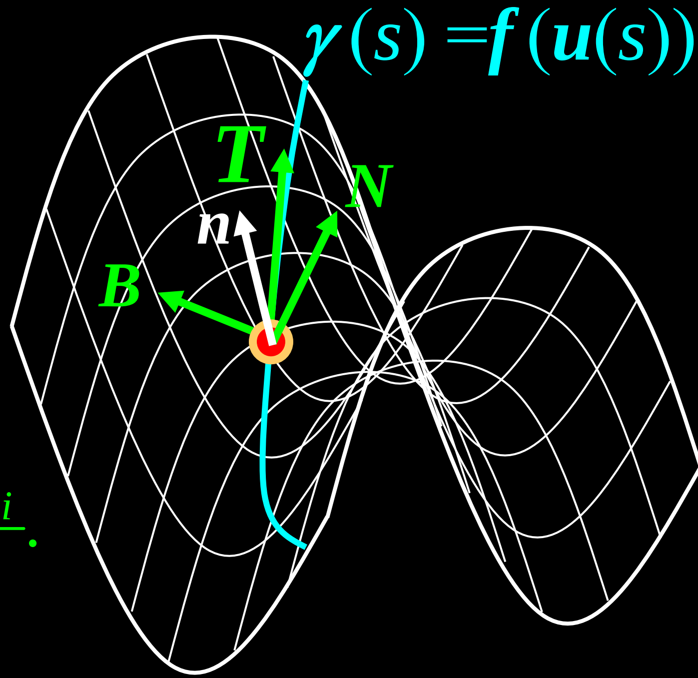
Second Fundamental Form (Cont.)

We already know that $\boldsymbol{T} = \gamma'(s) = \frac{\partial \boldsymbol{f}}{\partial u_1}\frac{du_1}{ds} + \frac{\partial \boldsymbol{f}}{\partial u_2}\frac{du_2}{ds}.$

Differentiating $\boldsymbol{T}$ to find the curvature vector $\kappa\boldsymbol{N}$:

$$\gamma''(s) = \frac{d}{ds}\sum_i \frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_i}{ds}$$

$$= \sum_i \frac{d}{ds}\frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_i}{ds} + \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i}\frac{d^2u_i}{ds^2}.$$



$$\gamma''(s) = \sum_i \frac{d}{ds}\frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_i}{ds} + \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i}\frac{d^2 u_i}{ds^2}.$$

Second Fundamental Form (Cont.)

where $$\frac{d}{ds}\frac{\partial \boldsymbol{f}}{\partial u_i} = \frac{\partial}{\partial u_1}\frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_1}{ds} + \frac{\partial}{\partial u_2}\frac{\partial \boldsymbol{f}}{\partial u_i}\frac{du_2}{ds}$$

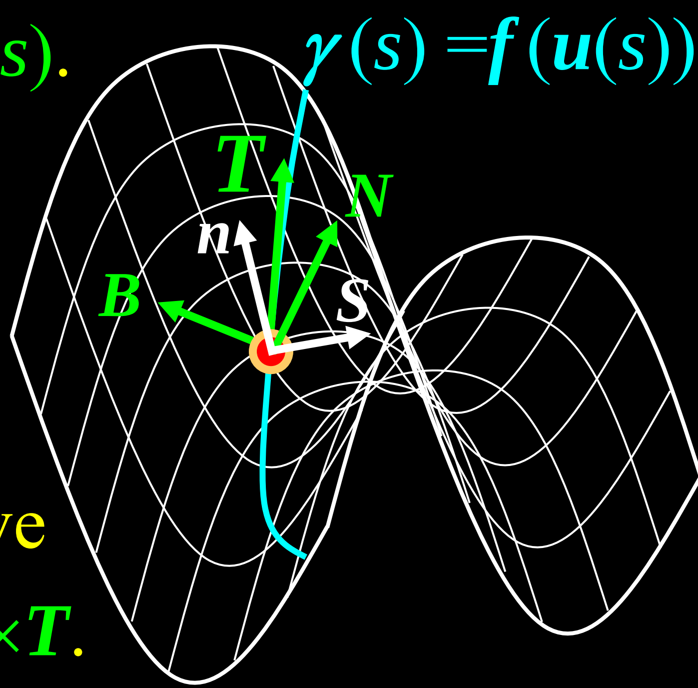
$$= \frac{\partial^2 \boldsymbol{f}}{\partial u_i \partial u_1}\frac{du_1}{ds} + \frac{\partial^2 \boldsymbol{f}}{\partial u_i \partial u_2}\frac{du_2}{ds}$$

$$= \sum_j \frac{\partial^2 \boldsymbol{f}}{\partial u_i \partial u_j}\frac{du_j}{ds}.$$

or $$\gamma''(s) = \sum_j \sum_i \frac{\partial^2 \boldsymbol{f}}{\partial u_j \partial u_i}\frac{du_j}{ds}\frac{du_i}{ds} + \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i}\frac{d^2 u_i}{ds^2} = \boldsymbol{v}(s) + \boldsymbol{t}(s).$$

Second Fundamental Form (Cont.)

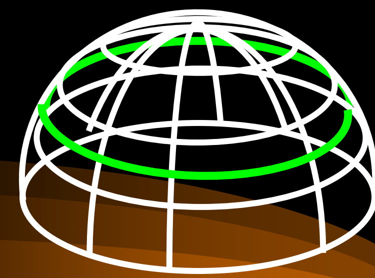
Recall that $\gamma''(s) = \kappa N$ or the curvature vector of $\gamma(s)$ has components in the direction of the tangent plane $t(s)$ and some other direction, $v(s)$.



Definition 12.34

For a curve γ in the surface f, the **intrinsic normal** to the curve at a point on the curve is $S = n \times T$.

Example 12.35



Consider curve, $\gamma(s) = \left(\frac{1}{\sqrt{2}} \cos\left(s\sqrt{2}\right), \frac{1}{\sqrt{2}} \sin\left(s\sqrt{2}\right), \frac{1}{\sqrt{2}} \right)$,

on hemisphere, $\boldsymbol{f}(\boldsymbol{u}) = \left(u_1, u_2, \sqrt{1 - (u_1)^2 - (u_2)^2} \right)$.

Then,

$$\gamma'(s) = \boldsymbol{T} = \left(-\sin\left(s\sqrt{2}\right), \cos\left(s\sqrt{2}\right), 0 \right),$$

$$\gamma''(s) = \boldsymbol{T}' = \kappa \boldsymbol{N} = \left(-\sqrt{2}\cos\left(s\sqrt{2}\right), -\sqrt{2}\sin\left(s\sqrt{2}\right), 0 \right).$$

or $\boldsymbol{N} = \left(-\cos\left(s\sqrt{2}\right), -\sin\left(s\sqrt{2}\right), 0 \right)$, and $\kappa = \sqrt{2}$.

Example 12.35 (Cont.)

Differentiating $\boldsymbol{f}(\boldsymbol{u})$, $\frac{\partial f}{\partial u_1} = \left(1, 0, \frac{-u_1}{\sqrt{1-(u_1)^2-(u_2)^2}} \right),$

and $\frac{\partial f}{\partial u_2} = \left(0, 1, \frac{-u_2}{\sqrt{1-(u_1)^2-(u_2)^2}} \right).$

The surface normal is $\boldsymbol{n} = \left(u_1, u_2, \sqrt{1-(u_1)^2-(u_2)^2} \right).$

Second Fundamental Form (Cont.)

Because $\boldsymbol{S} = \boldsymbol{n} \times \boldsymbol{T}$, $(\boldsymbol{S}, \boldsymbol{n}, \boldsymbol{T})$ forms an orthonormal system and one can write

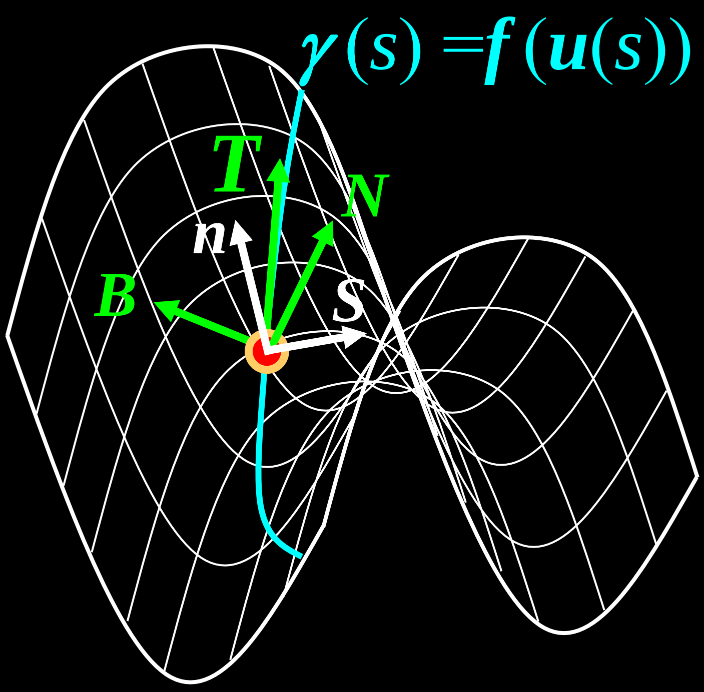
$$\gamma''(s) = \langle \gamma'', \boldsymbol{T} \rangle \boldsymbol{T} + \langle \gamma'', \boldsymbol{S} \rangle \boldsymbol{S} + \langle \gamma'', \boldsymbol{n} \rangle \boldsymbol{n}.$$

Because

$$\langle \gamma''(s), \boldsymbol{T} \rangle = \langle \kappa \boldsymbol{N}, \boldsymbol{T} \rangle = 0,$$

we have,

$$\gamma''(s) = \langle \gamma'', \boldsymbol{S} \rangle \boldsymbol{S} + \langle \gamma'', \boldsymbol{n} \rangle \boldsymbol{n}.$$



Definition 12.36

For an arc length regular parameterized curve $\gamma(s)$ in the simple surface $\boldsymbol{f}$, denote by

- $\kappa_n(s) = (\gamma``(s), \boldsymbol{n})$, the portion of the curvature vector of $\gamma(s)$ in the direction of the surface normal, called the **normal curvature** of $\gamma(s)$.
- $\kappa_g(s) = (\gamma``(s), \boldsymbol{S})$, the portion of the curvature vector of $\gamma(s)$ in the direction of the curve' intrinsic normal, called the **geodesic curvature** of $\gamma(s)$.

$$\gamma''(s) = \sum_j \sum_i \frac{\partial^2 \boldsymbol{f}}{\partial u_j \partial u_i} \frac{du_j}{ds} \frac{du_i}{ds} + \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i} \frac{d^2 u_i}{ds^2} = \boldsymbol{v}(s) + \boldsymbol{t}(s).$$

Second Fundamental Form (Cont.)

Because $\partial \boldsymbol{f} / \partial u_i$ is in the tangent plane of the surface, we see that $\kappa_n(s)\, \boldsymbol{n}$ is contributed from $\boldsymbol{v}(s)$ only, and also

$$\gamma``(s) = \kappa \boldsymbol{N} = \kappa_n(s)\, \boldsymbol{n} + \kappa_g(s)\, \boldsymbol{S}.$$

Further, because $\boldsymbol{n}$ and $\boldsymbol{S}$ are orthogonal,

$$\kappa^2 = (\kappa_n)^2 + (\kappa_g)^2.$$

Definition 12.37

A **geodesic curve** on the surface ***f*** is a unit speed regular curve in ***f*** with geodesic curvature κ_g equal to zero everywhere along the curve.

Note that when $\kappa_g = 0$, then $\boldsymbol{N} = \boldsymbol{n}$.

Question: What are the geodesics on a sphere?

$$\gamma''(s) = \sum_j \sum_i \frac{\partial^2 \boldsymbol{f}}{\partial u_j \partial u_i} \frac{du_j}{ds} \frac{du_i}{ds} + \sum_i \frac{\partial \boldsymbol{f}}{\partial u_i} \frac{d^2 u_i}{ds^2} = \boldsymbol{v}(s) + \boldsymbol{t}(s).$$

Second Fundamental Form (Cont.)

$$\begin{aligned}
\kappa_n &= \left\langle \gamma''(s), \boldsymbol{n} \right\rangle \\
&= \sum_j \sum_i \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_j \partial u_i}, \boldsymbol{n} \right\rangle \frac{du_j}{ds} \frac{du_i}{ds} + \sum_i \left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \boldsymbol{n} \right\rangle \frac{d^2 u_i}{ds^2} \\
&= \sum_j \sum_i L_{j,i} \frac{du_j}{ds} \frac{du_i}{ds} \\
&= \left[\frac{du_1}{ds} \; \frac{du_2}{ds} \right] \boldsymbol{L} \left[\frac{du_1}{ds} \; \frac{du_2}{ds} \right]^T .
\end{aligned}$$

Definition 12.38

The scalars $L_{i,j} = \left\langle \frac{\partial^2 \mathbf{f}}{\partial u_i \partial u_j}, \mathbf{n} \right\rangle$ are called the **coefficients of the second fundamental form**, and the matrix $\mathbf{L} = (L_{i,j})$ is called the **matrix of the second fundamental form**. The coefficients are also written $L = L_{1,1}$, $M = L_{1,2} = L_{2,1}$, and $N = L_{2,2}$.

Definition 12.38 (Cont.)

For an arbitrary tangent vector, $\left[\frac{\partial u_1}{\partial t} \; \frac{\partial u_2}{\partial t}\right]$,

the form

$$\mathrm{II}\left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right) = \left[\frac{\partial u_1}{\partial t} \; \frac{\partial u_2}{\partial t}\right] \boldsymbol{L} \left[\frac{\partial u_1}{\partial t} \; \frac{\partial u_2}{\partial t}\right]^T,$$

is called the **second fundamental form**.

Lemma 12.39

$$L_{i,j} = -\left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \frac{\partial \boldsymbol{n}}{\partial u_j} \right\rangle$$

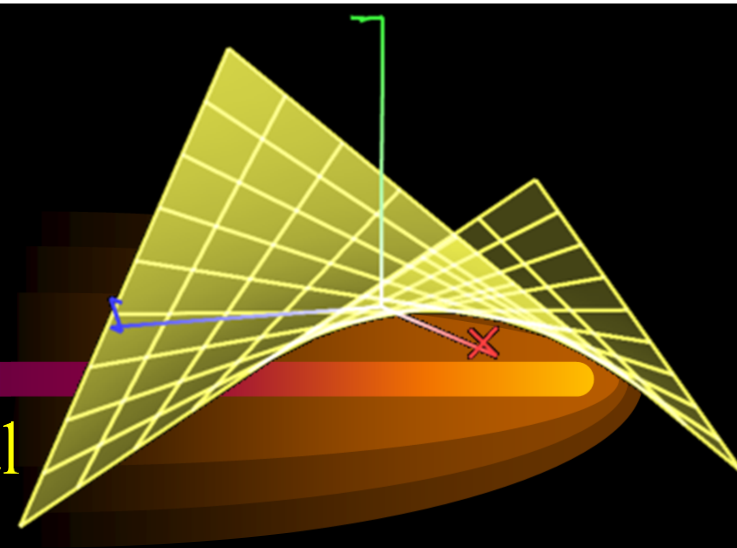
Proof:

Because $\left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \boldsymbol{n} \right\rangle \equiv 0$ for all values of $\boldsymbol{u}$,

$$0 = \frac{\partial}{\partial u_j} \left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \boldsymbol{n} \right\rangle = \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_i \partial u_j}, \boldsymbol{n} \right\rangle + \left\langle \frac{\partial \boldsymbol{f}}{\partial u_i}, \frac{\partial \boldsymbol{n}}{\partial u_j} \right\rangle$$

and the result follow.

Example 12.40



We calculate the second fundamental form's coefficients for surface $\boldsymbol{f}(\boldsymbol{u}) = (u_1+u_2,\ u_1u_2,\ u_1\text{-}u_2)$:

$$\frac{\partial \boldsymbol{f}}{\partial u_1} = (1, u_2, 1), \qquad \frac{\partial \boldsymbol{f}}{\partial u_2} = (1, u_1, -1),$$

$$\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} = (-u_1 - u_2, 2, u_1 - u_2),$$

and $$\boldsymbol{n} = \frac{(-u_1 - u_2, 2, u_1 - u_2)}{\sqrt{(u_1 + u_2)^2 + 4 + (u_1 - u_2)^2}}.$$

$$\frac{\partial f}{\partial u_1} = (1, u_2, 1), \qquad \frac{\partial f}{\partial u_2} = (1, u_1, -1)$$

Example 12.40 (Cont.)

$$\boldsymbol{n} = \frac{(-u_1 - u_2, 2, u_1 - u_2)}{\sqrt{(u_1 + u_2)^2 + 4 + (u_1 - u_2)^2}}.$$

Further,

$$\frac{\partial^2 \boldsymbol{f}}{\partial u_1{}^2} = (0,0,0), \qquad \frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2} = (0,1,0),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial u_2 \partial u_1} = (0,1,0), \qquad \frac{\partial^2 \boldsymbol{f}}{\partial u_2{}^2} = (0,0,0),$$

or

$$\boldsymbol{L} = \frac{2}{\sqrt{(u_1 + u_2)^2 + 4 + (u_1 - u_2)^2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Theorem 12.41

The second fundamental form is invariant under coordinate transformations which have a positive Jacobian.

Question: What would happen for a negative Jacobian?

Theorem 12.42

If $\gamma_1(s)$ and $\gamma_2(s)$ are two arc length parameterized curves in f with same tangent vector T at the same (intersection) surface point, p, they both have the same normal curvature at p.

Proof

$\kappa_n = \left[\frac{du_1}{ds} \; \frac{du_2}{ds}\right] L \; \left[\frac{du_1}{ds} \; \frac{du_2}{ds}\right]^T$ depends only on the coefficients of the second fundamental form and the tangent vector of the curves.

Theorem 12.43

For curve $\gamma(s)$, an arc length parameterized curve with normal $\boldsymbol{N}$ in surface $\boldsymbol{f}$, let $\boldsymbol{\theta}$ be the angle between $\boldsymbol{N}$ and $\boldsymbol{n}$. Then,

$$\begin{aligned}\kappa_n &= \left\langle \gamma''(s), \boldsymbol{n} \right\rangle \\ &= \kappa \left\langle \boldsymbol{N}, \boldsymbol{n} \right\rangle \\ &= \kappa \cos\theta.\end{aligned}$$

Back to the Second Fundamental Form

$$\kappa_n = L\left(\frac{du_1}{ds}\right)^2 + 2M\frac{du_1}{ds}\frac{du_2}{ds} + N\left(\frac{du_2}{ds}\right)^2$$

$$= \left[L\left(\frac{du_1}{dt}\right)^2 + 2M\frac{du_1}{dt}\frac{du_2}{dt} + N\left(\frac{du_2}{dt}\right)^2\right]\left(\frac{dt}{ds}\right)^2$$

$$= \frac{L\left(\frac{du_1}{dt}\right)^2 + 2M\frac{du_1}{dt}\frac{du_2}{dt} + N\left(\frac{du_2}{dt}\right)^2}{E\left(\frac{du_1}{dt}\right)^2 + 2F\frac{du_1}{dt}\frac{du_2}{dt} + G\left(\frac{du_2}{dt}\right)^2} = \frac{\mathrm{II}\left(\frac{d\mathbf{u}}{dt}\right)}{\mathrm{I}\left(\frac{d\mathbf{u}}{dt}\right)}.$$

Principal Curvatures

Consider the set S of all regular curves $\boldsymbol{u}(t) = (u_1(t), u_2(t))$ such that $||d\boldsymbol{u}/dt|| = 1$ at some surface point $\boldsymbol{a}$.

The normal curvature is a continuous function over the different directions of $d\boldsymbol{u}/dt$ at $\boldsymbol{a}$, which is closed and bounded. We seek the maximum and minimum of κ_n over the set of S.

Definition 12.45

The maximum and minimum values of the normal curvature are called **principal curvatures**.

The directions for which these values are attained are called **principal directions** of the surface. The principal directions are unit vectors.

Principal Curvatures (Cont.)

Assume κ_n is continuously differentiable as a function $d\mathbf{u}/dt$, and let $\mathbf{v} = d\mathbf{u}/dt$. The normal curvature varies as $\mathbf{v}$ rotates along the unit circle and hence κ_n is a function of this change, $\kappa_n(\mathbf{v})$. Differentiating:

$$\frac{\partial \kappa_n}{\partial v_1} = \frac{\frac{\partial \mathrm{II}}{\partial v_1}\mathrm{I} - \frac{\partial \mathrm{I}}{\partial v_1}\mathrm{II}}{\mathrm{I}^2}, \text{ and } \frac{\partial \kappa_n}{\partial v_2} = \frac{\frac{\partial \mathrm{II}}{\partial v_2}\mathrm{I} - \frac{\partial \mathrm{I}}{\partial v_2}\mathrm{II}}{\mathrm{I}^2}.$$

Principal Curvatures (Cont.)

Seeking the extremal values and because $\mathrm{I} \neq 0$ (why?),

$$0 = \frac{\partial \mathrm{II}}{\partial v_1} - \frac{\partial \mathrm{I}}{\partial v_1}\frac{\mathrm{II}}{\mathrm{I}} = \frac{\partial \mathrm{II}}{\partial v_1} - \kappa_n \frac{\partial \mathrm{I}}{\partial v_1}, \quad \text{and} \quad 0 = \frac{\partial \mathrm{II}}{\partial v_2} - \kappa_n \frac{\partial \mathrm{I}}{\partial v_2}.$$

$\mathrm{II} = Lv_1^2 + 2Mv_1v_2 + Nv_2^2$ and $\mathrm{I} = Ev_1^2 + 2Fv_1v_2 + Gv_2^2$

or
$$\frac{\partial \mathrm{II}}{\partial v_1} = 2Lv_1 + 2Mv_2, \qquad \frac{\partial \mathrm{II}}{\partial v_2} = 2Mv_1 + 2Nv_2,$$

$$\frac{\partial \mathrm{I}}{\partial v_1} = 2Ev_1 + 2Fv_2, \qquad \frac{\partial \mathrm{I}}{\partial v_2} = 2Fv_1 + 2Gv_2.$$

Principal Curvatures (Cont.)

Substituting these partials into the extremal functions,

$$Lv_1 + Mv_2 - \kappa_n(Ev_1 + Fv_2) = 0,$$
$$Mv_1 + Nv_2 - \kappa_n(Fv_1 + Gv_2) = 0,$$

or,

$$(L - \kappa_n E)v_1 + (M - \kappa_n F)v_2 = 0,$$
$$(M - \kappa_n F)v_1 + (N - \kappa_n G)v_2 = 0.$$

In matrix form: $$\begin{bmatrix} L - \kappa_n E & M - \kappa_n F \\ M - \kappa_n F & N - \kappa_n G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

$$\begin{bmatrix} L-\kappa_n E & M-\kappa_n F \\ M-\kappa_n F & N-\kappa_n G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Principal Curvatures (Cont.)

The determinant of the matrix must be zero (why?). Therefore, expanding this determinant one gets,

$$(EG-F^2)\kappa^2 - (GL+EN-2FM)\kappa + (LN-M^2) = 0,$$

while from the properties of quadratic functions we get,

$$\kappa_1\kappa_2 = \frac{LN-M^2}{EG-F^2} \quad \text{and} \quad \kappa_1+\kappa_2 = \frac{GL+EN-2FM}{EG-F^2}.$$

Definition 12.46

The quantity $\kappa_1 \kappa_2$ is called the Gaussian curvature and is typically denoted by K.

The quantity $(\kappa_1+\kappa_2) / 2$ is called the mean curvature and is typically denoted by H.

The Gaussian and the mean (almost) curvatures are invariants of the surface and are considered intrinsic properties.

Principal Curvatures (Cont.)

Going back to the equation $\begin{bmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,$

we also require that $v_1^2 + v_2^2 = 1$. Using the first equation in the above determinant, one gets, for $i = 1,2$:

$$v_2^i = -\frac{(L-\kappa_i E)v_1^i}{M-\kappa_i F}.$$

Question: Why did we select the first equation and not the second?

Principal Curvatures (Cont.)

Then, $1 = \left(v_1{}^i\right)^2 + \left(v_2{}^i\right)^2$

$$= \left(v_1{}^i\right)^2 + \left(\frac{\left(L - \kappa_i E\right)v_1{}^i}{M - \kappa_i F}\right)^2$$

$$= \left(v_1{}^i\right)^2 \left[1 + \left(\frac{L - \kappa_i E}{M - \kappa_i F}\right)^2\right]$$

$$= \left(v_1{}^i\right)^2 \left[\frac{\left(M - \kappa_i F\right)^2 + \left(L - \kappa_i E\right)^2}{\left(M - \kappa_i F\right)^2}\right].$$

Principal Curvatures (Cont.)

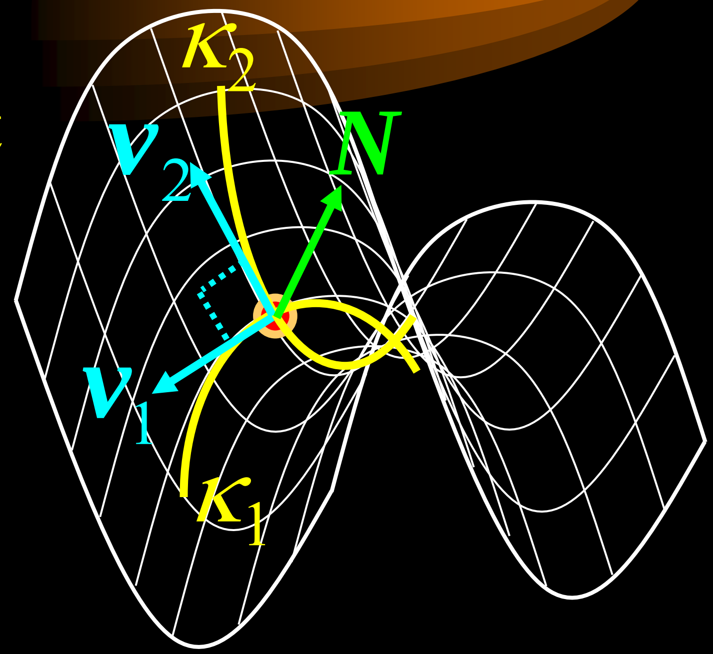
Hence, $v_1^{\,i} = \dfrac{M - \kappa_i F}{\sqrt{(M - \kappa_i F)^2 + (L - \kappa_i E)^2}}$,

and similarly,

$$v_2^{\,i} = -\frac{L - \kappa_i E}{\sqrt{(M - \kappa_i F)^2 + (L - \kappa_i E)^2}}.$$

Lemma 12.47

The principal directions at a point on a surface are orthogonal.



Proof

Let the two principal directions be $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$. Show that $\left\langle \boldsymbol{v}^1, \boldsymbol{v}^2 \right\rangle = \left[v_1^{\ 2} \ v_2^{\ 2} \right] \boldsymbol{G} \left[v_1^{\ 1} \ v_2^{\ 1} \right]^T = 0.$

Question: What if $\kappa_1 = \kappa_2$?

Example 12.48

Let $\boldsymbol{f}(\boldsymbol{u}) = (r \cos u_1 \cos u_2, r \sin u_1 \cos u_2, r \sin u_2)$, $r > 0$.

First order derivatives yield:

$$\frac{\partial \boldsymbol{f}}{\partial u_1} = \left(-r \sin u_1 \cos u_2, r \cos u_1 \cos u_2, 0\right),$$

$$\frac{\partial \boldsymbol{f}}{\partial u_2} = \left(-r \cos u_1 \sin u_2, -r \sin u_1 \sin u_2, r \cos u_2\right),$$

$$\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} = r^2 \cos u_2 \underbrace{\left(\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2\right)}_{\boldsymbol{n}}$$

Example 12.48 (Cont.)

$$\frac{\partial f}{\partial u_1} = (-r\sin u_1 \cos u_2, r\cos u_1 \cos u_2, 0),$$

$$\frac{\partial f}{\partial u_2} = (-r\cos u_1 \sin u_2, -r\sin u_1 \sin u_2, r\cos u_2),$$

$$\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} = r^2 \cos u_2 \underbrace{(\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)}_{\boldsymbol{n}}$$

Hence, we have,

$$g_{1,1} = \left\langle \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_1} \right\rangle = r^2 \cos^2 u_2,$$

$$g_{2,2} = \left\langle \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_2} \right\rangle = r^2, \quad \Rightarrow \quad \boldsymbol{G} = \begin{bmatrix} r^2 \cos^2 u_2 & 0 \\ 0 & r^2 \end{bmatrix}.$$

$$g_{1,2} = g_{2,1} = \left\langle \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2} \right\rangle$$

$$= r^2 (\sin u_1 \cos u_1 \sin u_2 \cos u_2 - \sin u_1 \cos u_1 \sin u_2 \cos u_2) = 0.$$

Example 12.48 (Cont.)

$$\frac{\partial f}{\partial u_1} = \left(-r\sin u_1 \cos u_2, r\cos u_1 \cos u_2, 0\right),$$

$$\frac{\partial f}{\partial u_2} = \left(-r\cos u_1 \sin u_2, -r\sin u_1 \sin u_2, r\cos u_2\right),$$

$$\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} = r^2 \cos u_2 \underbrace{(\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)}_{\boldsymbol{n}}$$

Second order derivatives yield,

$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_1}^2} = \left(-r\cos u_1 \cos u_2, -r\sin u_1 \cos u_2, 0\right),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2} = \left(r\sin u_1 \sin u_2, -r\cos u_1 \sin u_2, 0\right),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_2}^2} = \left(-r\cos u_1 \cos u_2, -r\sin u_1 \cos u_2, -r\sin u_2\right).$$

Example 12.48 (Cont.)

$$\frac{\partial^2 f}{\partial u_1^2} = (-r\cos u_1 \cos u_2, -r\sin u_1 \cos u_2, 0),$$
$$\frac{\partial^2 f}{\partial u_1 \partial u_2} = (r\sin u_1 \sin u_2, -r\cos u_1 \sin u_2, 0),$$
$$\frac{\partial^2 f}{\partial u_2^2} = (-r\cos u_1 \cos u_2, -r\sin u_1 \cos u_2, -r\sin u_2).$$

and recalling that $\boldsymbol{n} = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)$,

$$L_{1,1} = \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_1^2}, \boldsymbol{n} \right\rangle = -r\cos^2 u_2,$$

$$L_{1,2} = \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2}, \boldsymbol{n} \right\rangle = 0, \quad \Rightarrow \quad \boldsymbol{L} = \begin{bmatrix} -r\cos^2 u_2 & 0 \\ 0 & -r \end{bmatrix}.$$

$$L_{2,2} = \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_2^2}, \boldsymbol{n} \right\rangle = -r.$$

$$\kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{|\boldsymbol{L}|}{|\boldsymbol{G}|} \qquad \boldsymbol{L} = \begin{bmatrix} -r\cos^2 u_2 & 0 \\ 0 & -r \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} r^2\cos^2 u_2 & 0 \\ 0 & r^2 \end{bmatrix}$$

Example 12.48 (Cont.)

The Gaussian curvature equals:

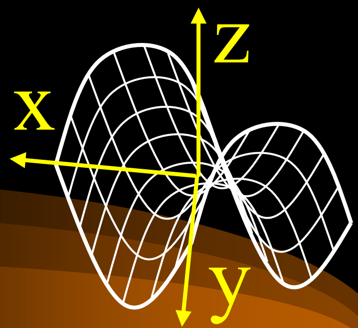
$$K = \kappa_1\kappa_2 = \frac{|\boldsymbol{L}|}{|\boldsymbol{G}|} = \frac{r^2\cos^2 u_2}{r^4\cos^2 u_2} = \frac{1}{r^2},$$

and the mean curvature equals:

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{GL + EN\text{-}2FM}{2|\boldsymbol{G}|} = \frac{g_{2,2}L_{1,1} + g_{1,1}L_{2,2} - 2g_{1,2}L_{1,2}}{2|\boldsymbol{G}|}$$

$$= \frac{-r^3\cos^2 u_2 - r^3\cos^2 u_2}{2r^4\cos^2 u_2} = -\frac{1}{r}.$$

Example 12.49



A saddle surface with parameterization $\boldsymbol{f}(\boldsymbol{u}) = (u_1, u_2, u_1 u_2)$.

First order derivatives yield:

$$\frac{\partial \boldsymbol{f}}{\partial u_1} = (1, 0, u_2), \quad \frac{\partial \boldsymbol{f}}{\partial u_2} = (0, 1, u_1), \quad \Rightarrow \quad \boldsymbol{G} = \begin{bmatrix} 1 + (u_2)^2 & u_1 u_2 \\ u_1 u_2 & 1 + (u_1)^2 \end{bmatrix}.$$

$$\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} = (-u_2, -u_1, 1) \quad \Rightarrow \quad \boldsymbol{n} = \frac{(-u_2, -u_1, 1)}{\sqrt{(u_1)^2 + (u_2)^2 + 1}}.$$

Example 12.49 (Cont.)

$$\boldsymbol{n} = \frac{(-u_2, -u_1, 1)}{\sqrt{(u_1)^2 + (u_2)^2 + 1}}$$

A saddle surface with parameterization $\boldsymbol{f}(\boldsymbol{u}) = (u_1, u_2, u_1 u_2)$.

Second order derivatives yield :

$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_1}^2} = (0,0,0),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2} = (0,0,1), \quad \Rightarrow \quad \boldsymbol{L} = \frac{1}{\sqrt{(u_1)^2 + (u_2)^2 + 1}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_2}^2} = (0,0,0).$$

Example 12.49 (Cont.)

$$L = \frac{1}{\sqrt{(u_1)^2 + (u_2)^2 + 1}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 + (u_2)^2 & u_1 u_2 \\ u_1 u_2 & 1 + (u_1)^2 \end{bmatrix}$$

The Gaussian curvature equals:

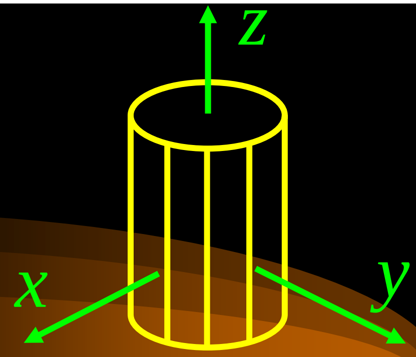
$$K = \kappa_1 \kappa_2 = \frac{|L|}{|G|} = -\frac{1}{\left(1 + (u_1)^2 + (u_2)^2\right)^2}.$$

and the mean curvature equals:

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{GL + EN\text{-}2FM}{2|G|} = \frac{g_{2,2} L_{1,1} + g_{1,1} L_{2,2} - 2 g_{1,2} L_{1,2}}{2|G|}$$

$$= -\frac{u_1 u_2}{\left(1 + (u_1)^2 + (u_2)^2\right)^{3/2}}.$$

Example 12.50



A cylindrical surface with radius r, $\boldsymbol{f}(\boldsymbol{u}) = (r \cos u_1, r \sin u_1, u_2)$. First order derivatives yield:

$$\frac{\partial \boldsymbol{f}}{\partial u_1} = (-r \sin u_1, r \cos u_1, 0), \qquad \frac{\partial \boldsymbol{f}}{\partial u_2} = (0,0,1), \quad \Rightarrow \quad \boldsymbol{G} = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\frac{\partial \boldsymbol{f}}{\partial u_1} \times \frac{\partial \boldsymbol{f}}{\partial u_2} = (r \cos u_1, r \sin u_1, 0) \quad \Rightarrow \quad \boldsymbol{n} = (\cos u_1, \sin u_1, 0).$$

Example 12.50 (Cont.)

$$\boldsymbol{n} = (\cos u_1, \sin u_1, 0)$$

Second order derivatives yield :

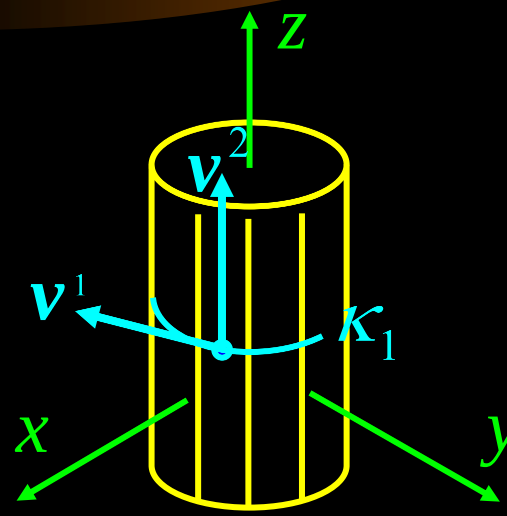
$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_1}^2} = (-r\cos u_1, -r\sin u_1, 0),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2} = (0,0,0), \qquad \Rightarrow \qquad \boldsymbol{L} = \begin{bmatrix} -r & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial {u_2}^2} = (0,0,0).$$

Example 12.50 (Cont.)

$$\boldsymbol{G} = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{L} = \begin{bmatrix} -r & 0 \\ 0 & 0 \end{bmatrix}$$



The Gaussian curvature equals:

$$K = \kappa_1 \kappa_2 = \frac{|\boldsymbol{L}|}{|\boldsymbol{G}|} = \frac{0}{r^2} = 0.$$

and the mean curvature equals:

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{GL + EN\text{-}2FM}{2|\boldsymbol{G}|}$$

$$= \frac{g_{2,2}L_{1,1} + g_{1,1}L_{2,2} - 2g_{1,2}L_{1,2}}{2|\boldsymbol{G}|} = -\frac{r}{2r^2} = -\frac{1}{2r}.$$

The Osculating Paraboloid

Consider the second order Taylor approximation to $\boldsymbol{f}$ at $\boldsymbol{a}$:

$$\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{v})-\boldsymbol{f}(\boldsymbol{a})=\left(\frac{\partial \boldsymbol{f}}{\partial u_1}(\boldsymbol{a})v_1+\frac{\partial \boldsymbol{f}}{\partial u_2}(\boldsymbol{a})v_2\right)$$
$$+\frac{1}{2}\left(\frac{\partial^2 \boldsymbol{f}}{\partial {u_1}^2}(\boldsymbol{a})(v_1)^2+2\frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2}(\boldsymbol{a})v_1v_2+\frac{\partial^2 \boldsymbol{f}}{\partial {u_2}^2}(\boldsymbol{a})(v_2)^2\right)$$
$$+R(\boldsymbol{v}).$$

where $R(\boldsymbol{v})$ is the remainder term of third order.

The Osculating Paraboloid (Cont.)

Define the second order approximation function around $\boldsymbol{a}$:

$$\delta_{\boldsymbol{f},\boldsymbol{a}}(\boldsymbol{v}) = \left(\frac{\partial \boldsymbol{f}}{\partial u_1}(\boldsymbol{a}) v_1 + \frac{\partial \boldsymbol{f}}{\partial u_2}(\boldsymbol{a}) v_2 \right) + \frac{1}{2}\left(\frac{\partial^2 \boldsymbol{f}}{\partial u_1^{\ 2}}(\boldsymbol{a})(v_1)^2 + 2\frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2}(\boldsymbol{a}) v_1 v_2 + \frac{\partial^2 \boldsymbol{f}}{\partial u_2^{\ 2}}(\boldsymbol{a})(v_2)^2 \right).$$

The first term is clearly in the tangent plane.

The Osculating Paraboloid (Cont.)

Considering the behavior of $\delta_{\boldsymbol{f},\boldsymbol{a}}(\boldsymbol{v})$ outside the tangent plane, we get,

$$\rho(\boldsymbol{v}) = \left\langle \delta_{\boldsymbol{f},\boldsymbol{a}}(\boldsymbol{v}), \boldsymbol{n} \right\rangle$$

$$= \frac{1}{2}\left(\left\langle \frac{\partial^2 \boldsymbol{f}}{\partial {u_1}^2}, \boldsymbol{n} \right\rangle (\boldsymbol{a}) (v_1)^2 + 2 \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial u_1 \partial u_2}, \boldsymbol{n} \right\rangle (\boldsymbol{a}) v_1 v_2 + \left\langle \frac{\partial^2 \boldsymbol{f}}{\partial {u_2}^2}, \boldsymbol{n} \right\rangle (\boldsymbol{a}) (v_2)^2 \right)$$

$$= \frac{1}{2}\left(L_{1,1}(v_1)^2 + 2L_{1,2} v_1 v_2 + L_{2,2}(v_2)^2 \right)$$

$$= \frac{1}{2}\,\mathrm{II}(v_1, v_2).$$

Definition 12.51

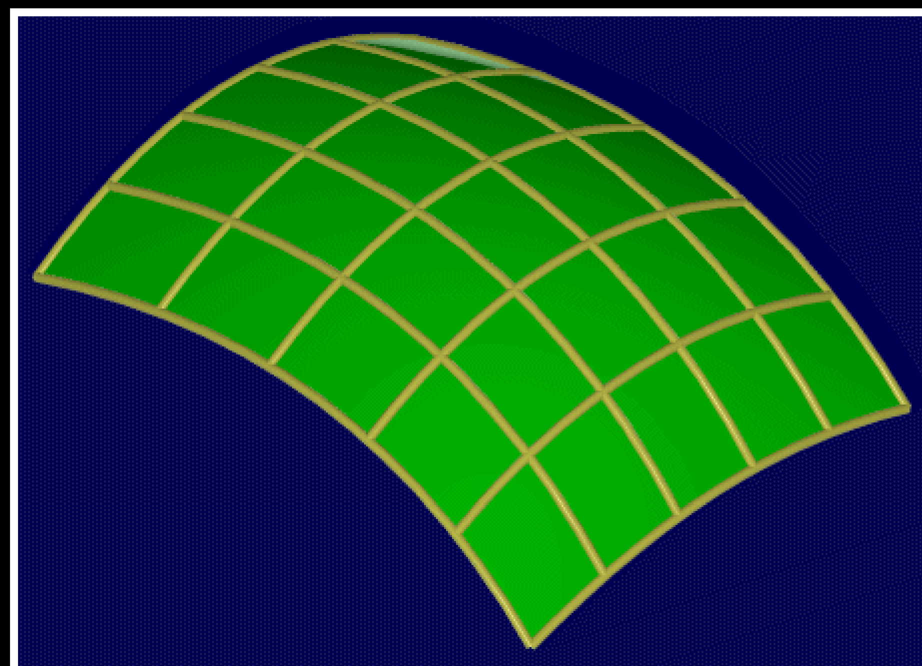
The surface $\rho(\mathbf{v})$, as a function of $\mathbf{v}$ measures the approximate distance of $\boldsymbol{f}$ from the tangent plane and is called the osculating paraboloid.

For each fixed value $\rho(\mathbf{v}) = \rho_0$, the resulting implicit curve is a conic, quadratic curve in $\mathbf{v} = (v_1, v_2)$.

The Osculating Paraboloid (Cont.)

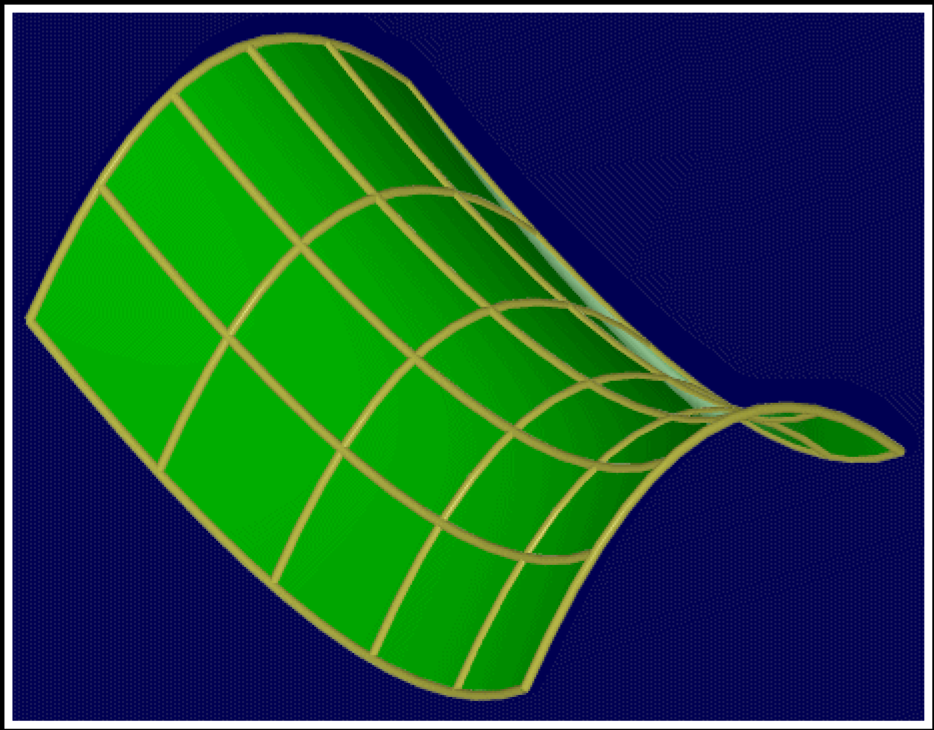
If $|\boldsymbol{L}| > 0$, the surface $\rho(\boldsymbol{v})$ is called an elliptic paraboloid.

Contours $\rho(\boldsymbol{v}) = \rho_0$ of the osculating paraboloid that are parallel to the tangent plane are ellipses.



The Osculating Paraboloid (Cont.)

If $|\boldsymbol{L}| < 0$, the surface $\rho(\boldsymbol{v})$ is called a hyperbolic paraboloid.

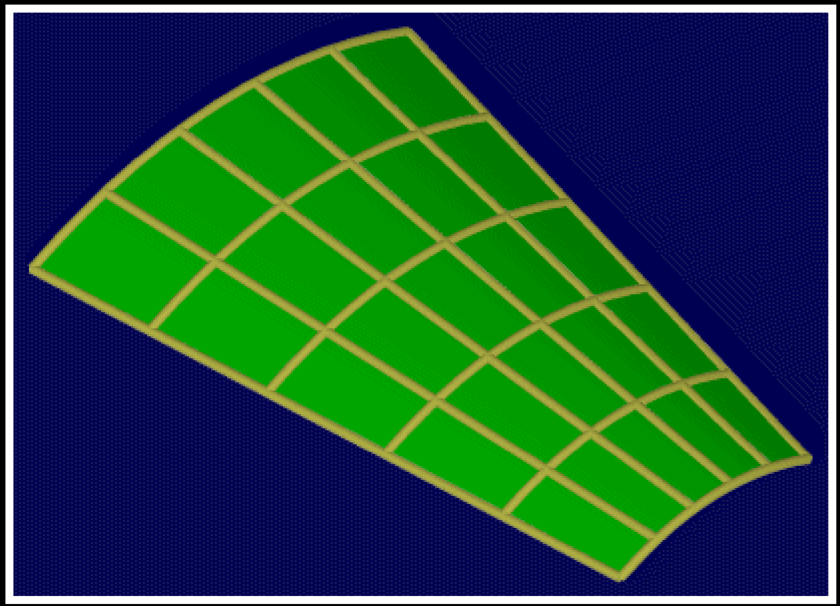


Contours $\rho(\boldsymbol{v}) = \rho_0$ of the osculating paraboloid are hyperbolas.

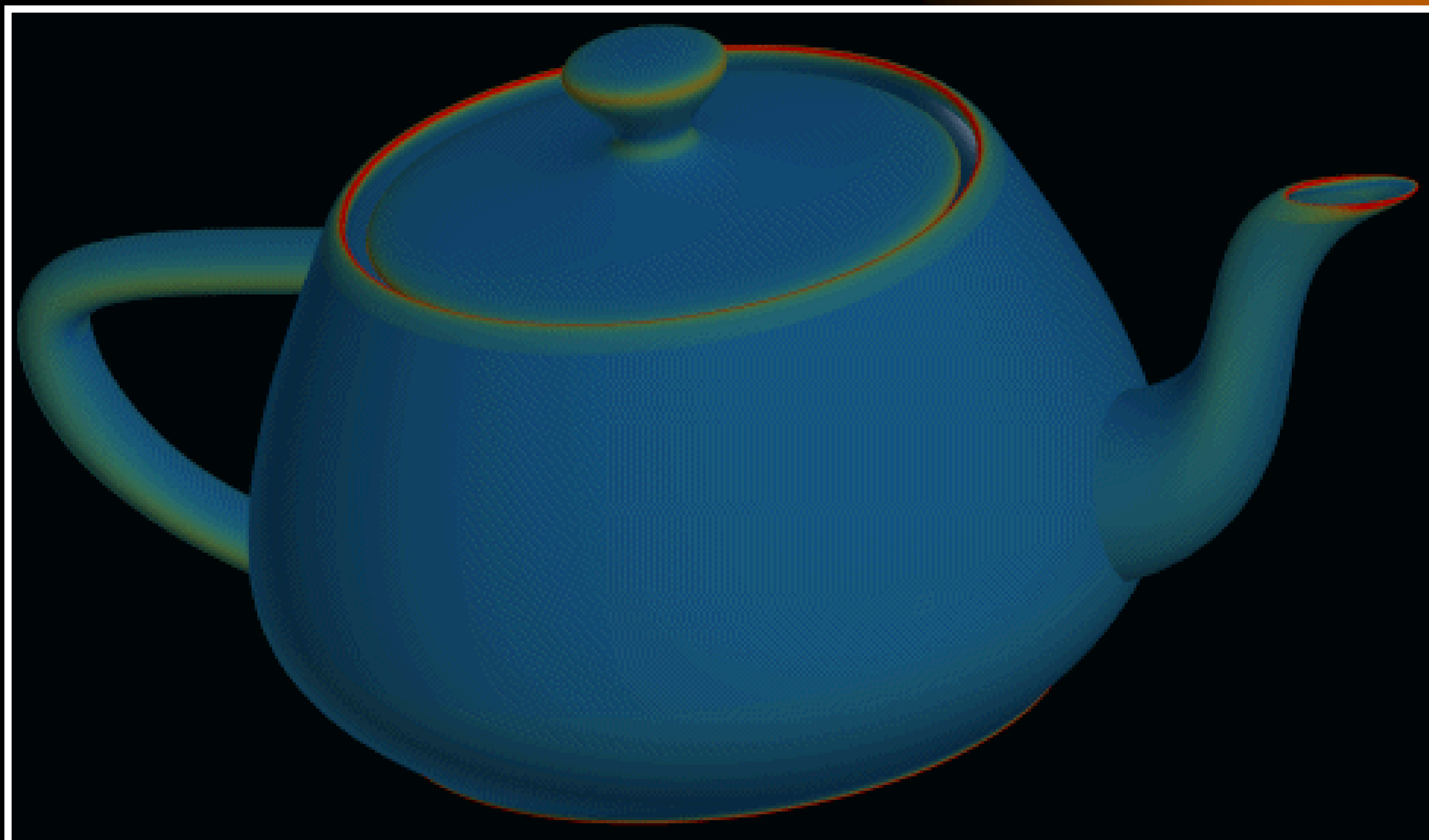
The Osculating Paraboloid (Cont.)

If $|\boldsymbol{L}| = 0$, the surface $\rho(\boldsymbol{v})$ is a parabolic cylinder.

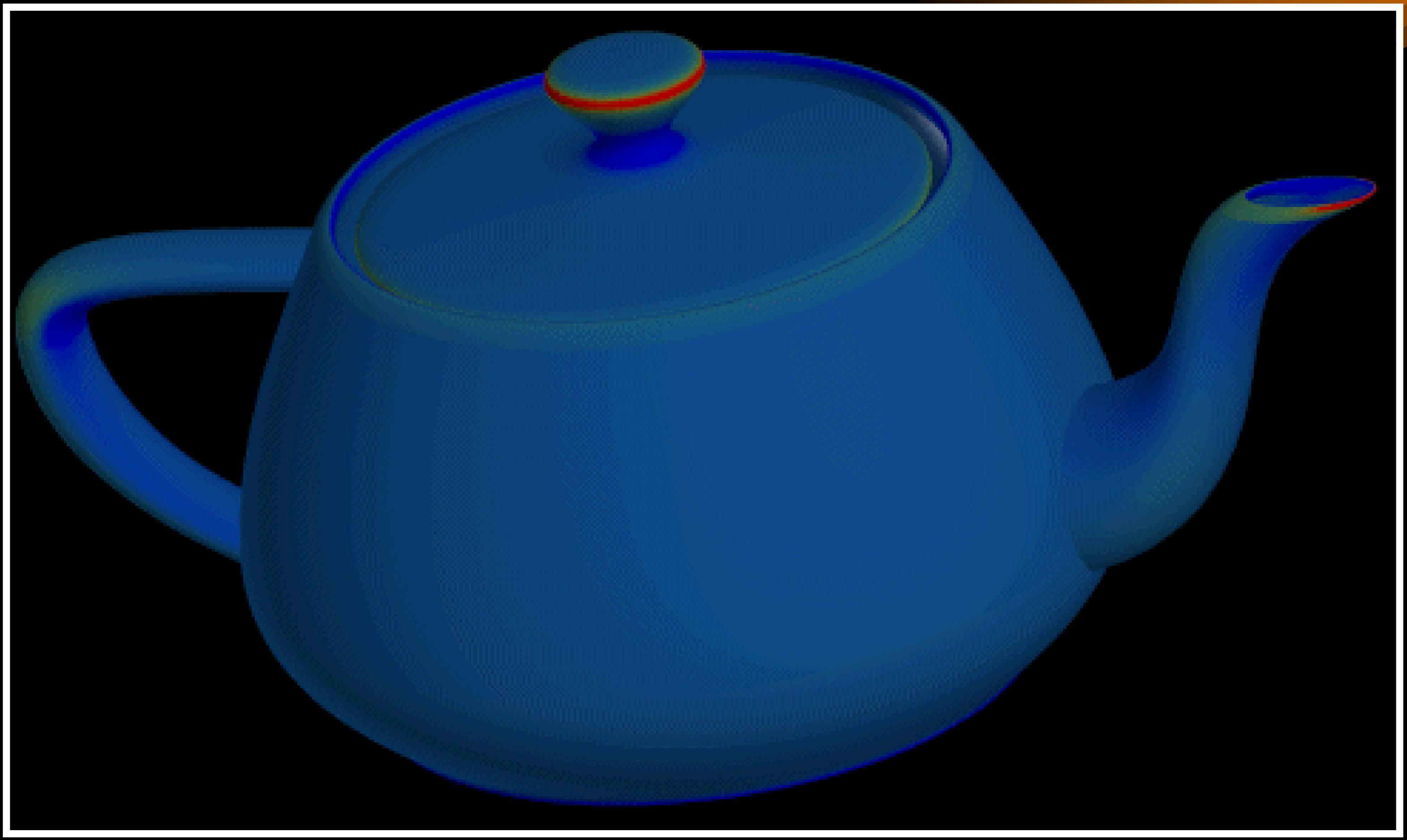
Contours $\rho(\boldsymbol{v}) = \rho_0$ of the osculating paraboloid are parabolas (or lines).



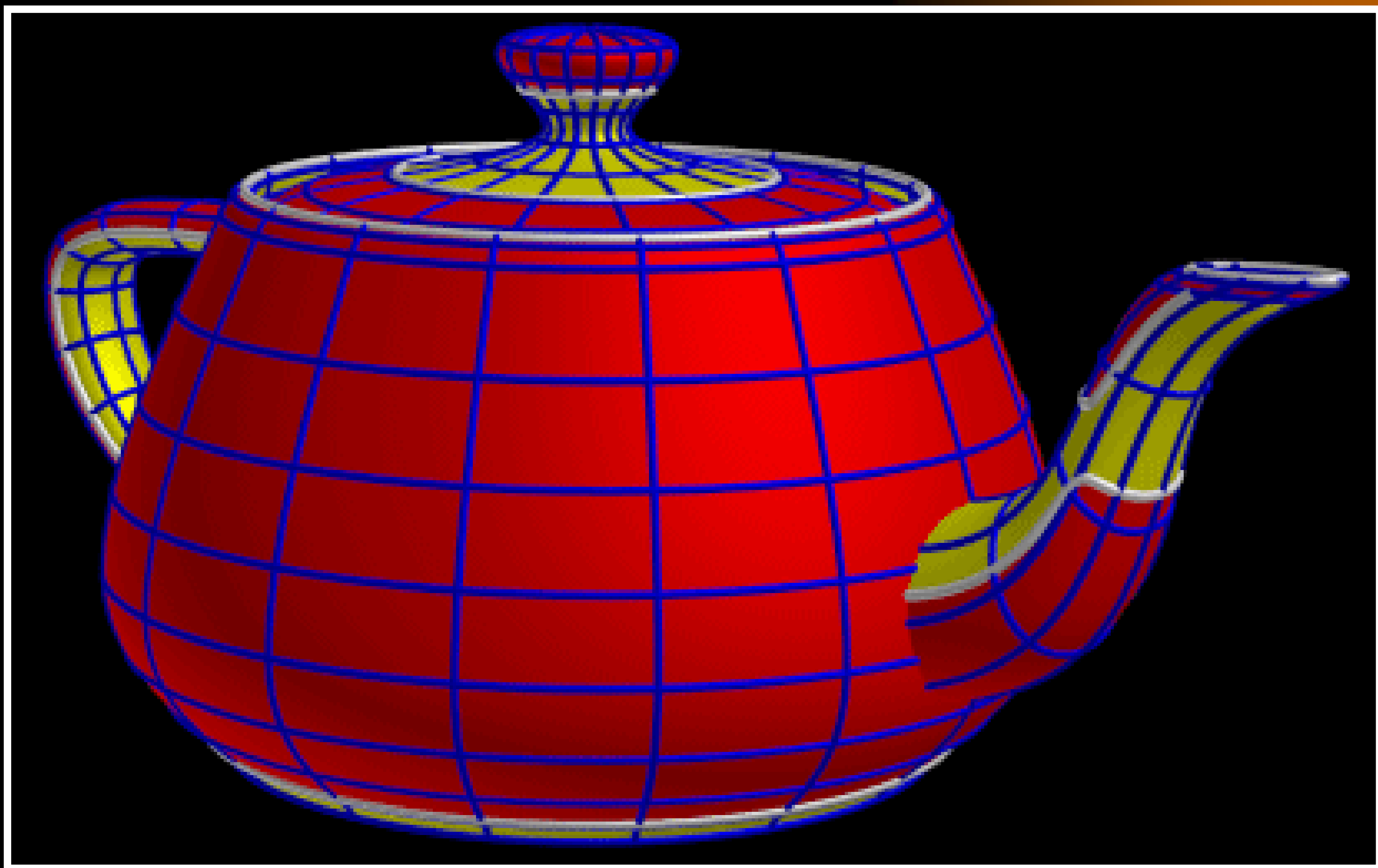
Teapot - Mean Curvature



Teapot - Gaussian Curvature



Teapot - Parabolic Edges



Polygonal Models – Curvature Estimation I

- There are many polygonal models out there
- In many cases, the models approximate C^2 smooth surfaces.
- A possible solution: to each vertex of the polygonal model, fit a paraboloid to the local neighborhood.
- Extract the principal curvatures (and hence K and H) and principal directions by examining the paraboloid.

Polygonal Models – Curvature Estimation II

The Gaussian Curvature at a vertex V, K_v, can also be estimated using vertices' angular deficiency:

$$K_v = \frac{2\pi - \sum_i \alpha_i^v}{A_v},$$

where A_v is the effective area associated with V, and α_i^v is the *ith* angle around vertex V.

Polygonal Models – Curvature Estimation III

The Gauss Bonnet theorem over a closed sufficiently continuous surface S states that

$$\int_s K = 2\pi\chi_s$$

where

$\chi_s = 2 - 2g$ is the Euler characteristics of the surface and g is its Genus.

$$\int_S K = 2\pi\chi_s$$

Polygonal Models – Curvature Estimation IV

And the discrete Gauss Bonnet theorem (triangular model) states that:

For V vertices, E edges and T triangles (and $2|E| = 3|T|$).

$$\sum_{v\in V} \boldsymbol{K}_v = \sum_{v\in V}\left(2\pi - \sum_i \alpha_i^v\right)$$

$$= 2\pi|V| - \sum_{v\in V}\sum_i \alpha_i^v$$

$$= 2\pi|V| - \pi|T|$$

$$= 2\pi|V| - \pi\left(2|E| - 2|T|\right)$$

$$= 2\pi\left(|V| - |E| + |T|\right)$$

$$= 2\pi\chi_s.$$