Computer Aided Geometric Design

Surface Representations

©Gershon Elber, Technion

based on a book by Cohen, Riesenfeld, & Elber

Tensor Product Surfaces

Definition 13.1:

Consider $\boldsymbol{F}$ and $\boldsymbol{G}$, two sets of univariate functions with intervals domains U and V, respectively,

$$\boldsymbol{F} = \{ f_i(u) \}_{i=0,m}, \quad \boldsymbol{G} = \{ g_j(v) \}_{j=0,n}.$$

A surface formed by

$$h(u,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} c_{i,j} f_i(u) g_j(v)$$

is called a tensor product surface with domain $U \times V$.

If $c_{i,j} \in R^3$ for all i, j, then h is a parametric surface.

A Bilinear Surface

Example 13.2:

Consider the linear blending functions

$\boldsymbol{F} = \{ f_0(u) = 1\text{-}u, \;\; f_1(u) = u \}$ and $\boldsymbol{G} = \{ g_0(v) = 1\text{-}v, \;\; g_1(v) = v \}$,

with domain $U = [0, 1]$ and $V = [0, 1]$. The tensor product surface

$$h(u,v) = c_{0,0} f_0(u) g_0(v) + c_{0,1} f_0(u) g_1(v)$$
$$+ c_{1,0} f_1(u) g_0(v) + c_{1,1} f_1(u) g_1(v)$$

is a bilinear tensor product surface with domain $U \times V$.

Question: What is the value of $h(u, v)$ at $u = 0, 1$ and $v = 0, 1$?

A Tensor Product Bezier Surface

Definition 13.3:

Consider $\boldsymbol{P} = \{ P_{i,j} \in R^3 \mid 0 \le i \le m ,\ 0 \le j \le n \}$ and collection of functions $\boldsymbol{F} = \{\theta_{i,m}(u)\}_{i=0}^{m}$ $\boldsymbol{G} = \{\theta_{j,n}(v)\}_{j=0}^{n}$.

The parametric surface

$$\sigma(u,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j}\theta_{i,m}(u)\theta_{j,n}(v)$$

is called a degree $m \times n$ tensor product Bezier surface with domain $U \times V = [0, 1] \times [0, 1]$.

Question: What is the value of $\sigma(u, v)$ at $u = 0, 1$ or $v = 0, 1$?

Bezier Surfaces (Cont.)

From properties of Bernstein polynomials

$$\sigma(u,0) = \sum_{j=0}^{n}\sum_{i=0}^{m} P_{i,j}\theta_{i,m}(u)\theta_{j,n}(0) = \sum_{i=0}^{m} P_{i,0}\theta_{i,m}(u),$$

$$\sigma(u,1) = \sum_{j=0}^{n}\sum_{i=0}^{m} P_{i,j}\theta_{i,m}(u)\theta_{j,n}(1) = \sum_{i=0}^{m} P_{i,n}\theta_{i,m}(u),$$

$$\sigma(0,v) = \sum_{j=0}^{n}\sum_{i=0}^{m} P_{i,j}\theta_{i,m}(0)\theta_{j,n}(v) = \sum_{j=0}^{n} P_{0,j}\theta_{j,n}(v),$$

$$\sigma(1,v) = \sum_{j=0}^{n}\sum_{i=0}^{m} P_{i,j}\theta_{i,m}(1)\theta_{j,n}(v) = \sum_{j=0}^{n} P_{m,j}\theta_{j,n}(v).$$

A Tensor Product B-spline Surface

Definition 13.4:

Consider $\boldsymbol{P} = \{ P_{i,j} \in R^3 \mid 0 \le i \le m ,\ 0 \le j \le n \}$ and collection of functions $\boldsymbol{F} = \{B_{i,k_u,\boldsymbol{\tau}_u}(u)\}_{i=0}^{m}$, $\boldsymbol{G} = \{B_{j,k_v,\boldsymbol{\tau}_v}(v)\}_{j=0}^{n}$, where $\boldsymbol{\tau}_u$ and $\boldsymbol{\tau}_v$ are knot vectors of length $m + k_u + 2$ and $n + k_v + 2$, respectively.

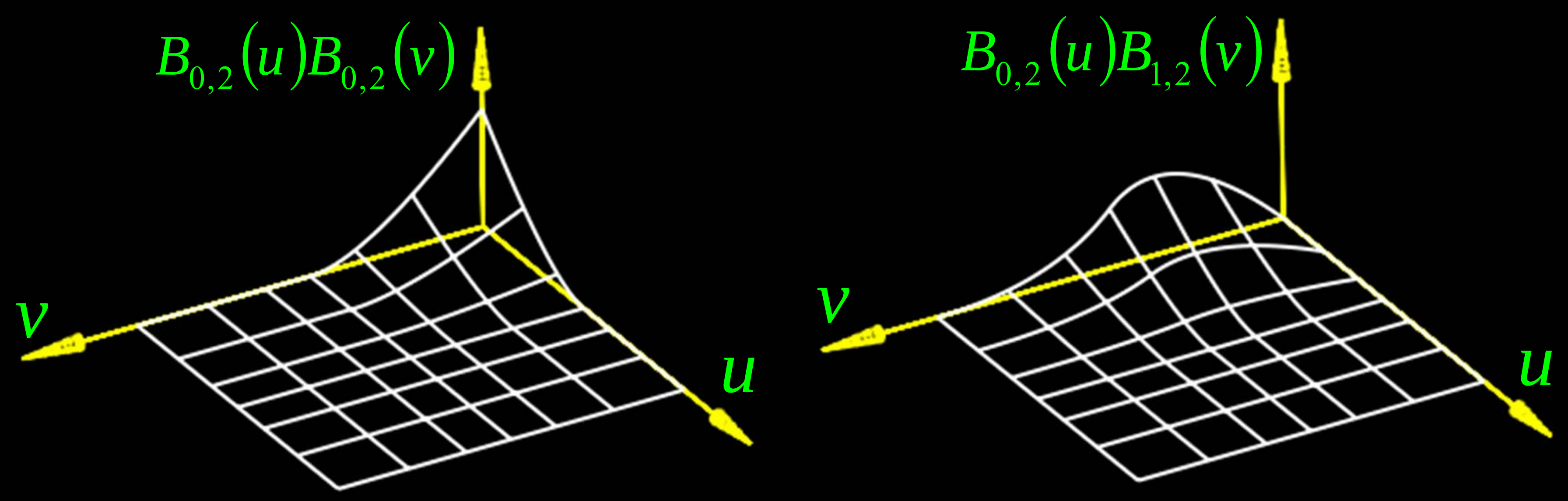
The parametric surface

$$\sigma(u,v) = \sum_{j=0}^{n}\sum_{i=0}^{m} P_{i,j} B_{i,k_u,\boldsymbol{\tau}_u}(u) B_{j,k_v,\boldsymbol{\tau}_v}(v)$$

is called a degree $k_u \times k_v$ tensor product B-spline surface with domain $U \times V = [\boldsymbol{\tau}_u(k_u) ,\ \boldsymbol{\tau}_u(m+1)) \times [\boldsymbol{\tau}_v(k_v) ,\ \boldsymbol{\tau}_v(n+1))$.

B-spline Products

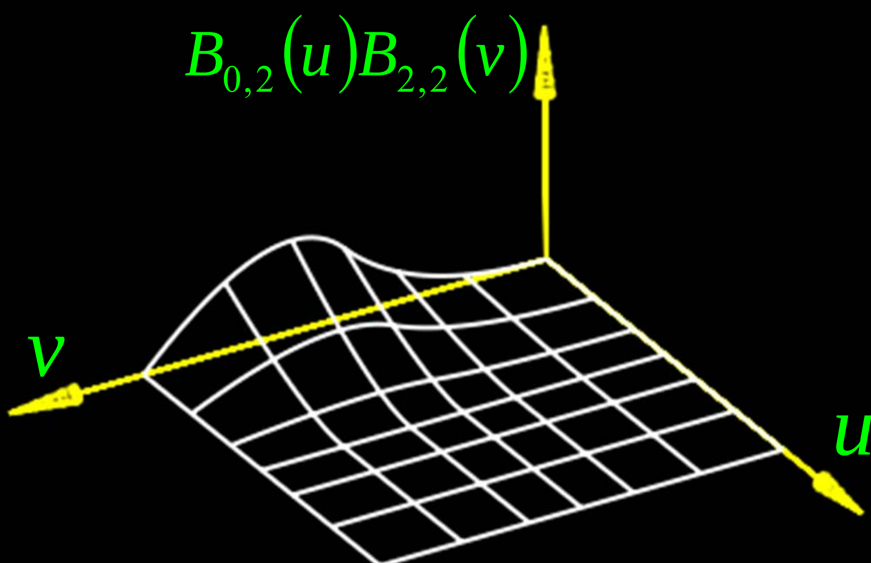
Example 13.5: Consider $\tau_u = \tau_v = \{0, 0, 0, 1, 2, 2, 2\}$ and quadratic functions. We draw the different products:

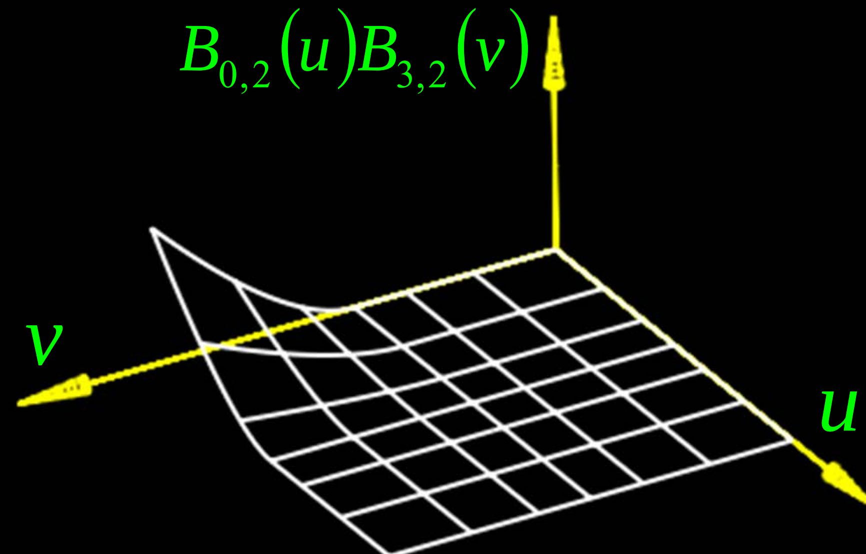


B-spline Products (Cont.)

Example 13.5 (Cont.):

$\tau_u = \tau_v = \{ 0, 0, 0, 1, 2, 2, 2\}.$

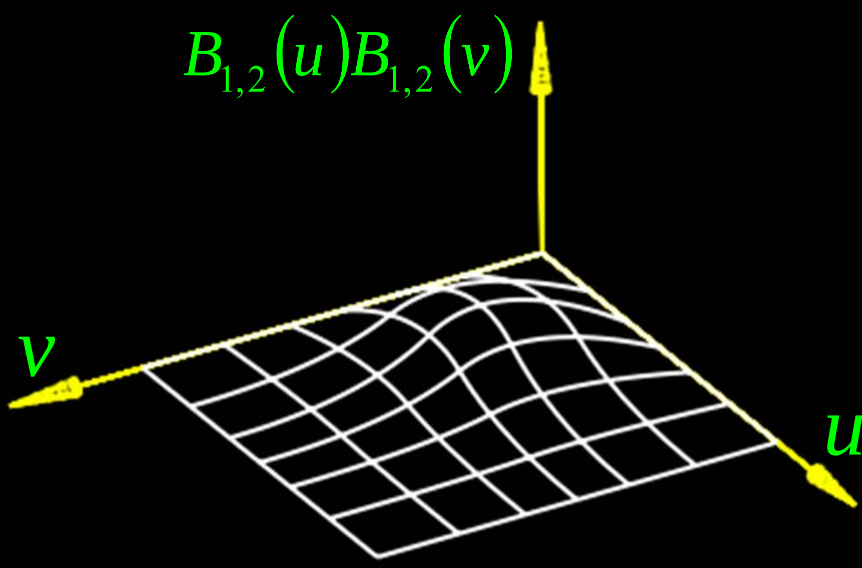


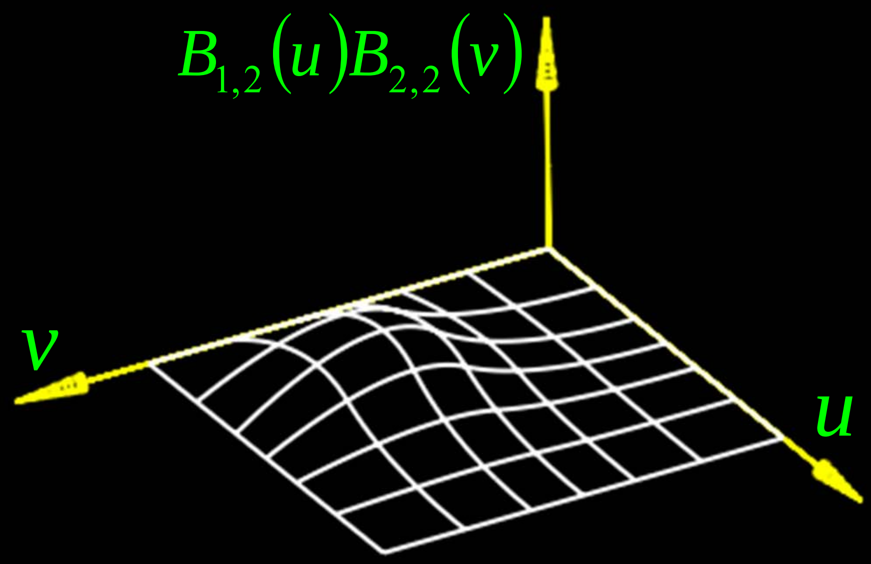


B-spline Products (Cont.)

Example 13.5 (Cont.):

$\tau_u = \tau_v = \{ 0, 0, 0, 1, 2, 2, 2 \}.$

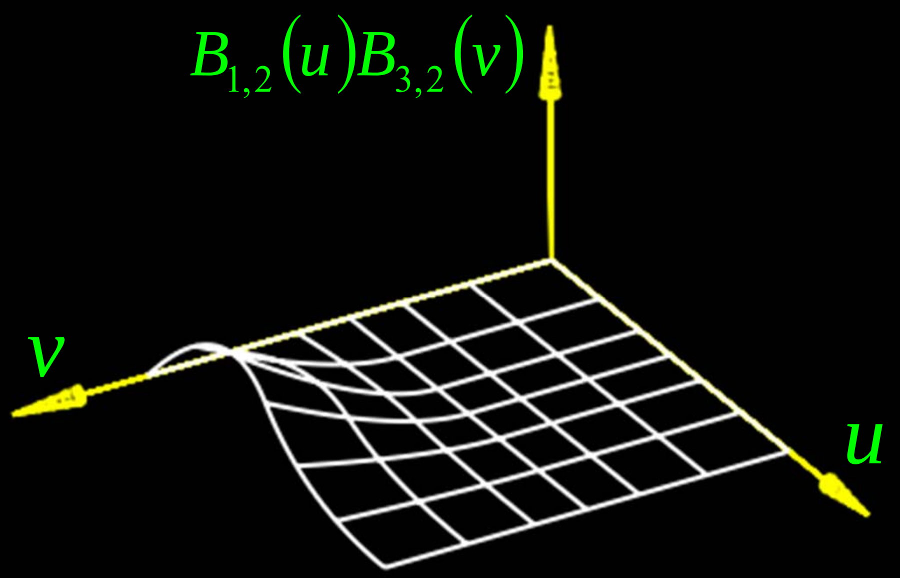


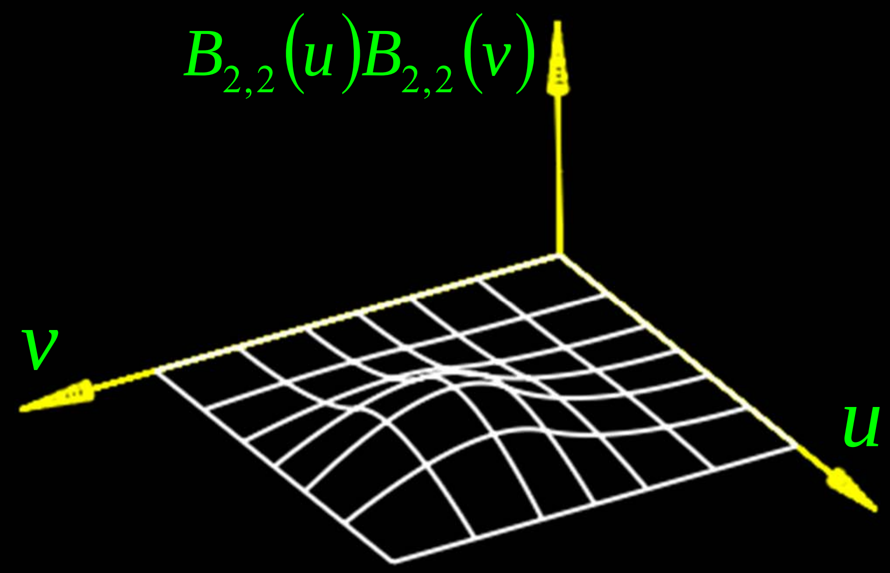


B-spline Products (Cont.)

Example 13.5 (Cont.):

$\tau_u = \tau_v = \{ 0, 0, 0, 1, 2, 2, 2\}.$

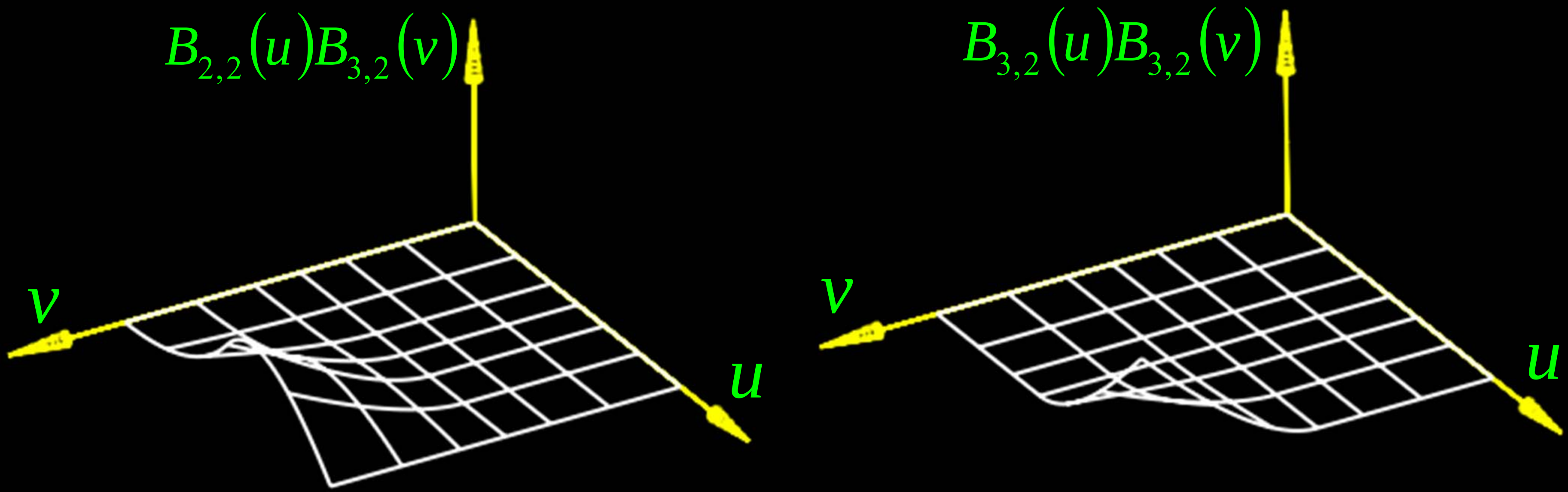




B-spline Products (Cont.)

Example 13.5 (Cont.):

$\tau_u = \tau_v = \{\, 0, 0, 0, 1, 2, 2, 2 \}.$



Control Mesh

Definition 13.7:

The collection

$$\boldsymbol{P} = \{ P_{i,j} \in R^3 \mid 0 \le i \le m\ ,\ \ 0 \le j \le n \}$$

is called the control mesh for the Bezier/B-spline surface.

Convex Combination

Lemma 13.8:

Suppose $\sigma(u,v)=\sum_{j=0}^{n}\sum_{i=0}^{m}P_{i,j}B_{i,k_u,\boldsymbol{\tau}_u}(u)B_{j,k_v,\boldsymbol{\tau}_v}(v)$ is a tensor product B-spline surface, for the proper domain. Then

$$1=\sum_{j=0}^{n}\sum_{i=0}^{m}B_{i,k_u,\boldsymbol{\tau}_u}(u)B_{j,k_v,\boldsymbol{\tau}_v}(v).$$

Convex Combination (Cont.)

Lemma 13.8 (cont.):

Proof: $\sum_{i=0}^{m} B_{i,k_u,\boldsymbol{\tau}_u}(u) = 1, \quad \sum_{j=0}^{n} B_{j,k_v,\boldsymbol{\tau}_v}(v) = 1.$

Then
$$\sum_{j=0}^{n}\sum_{i=0}^{m} B_{i,k_u,\boldsymbol{\tau}_u}(u) B_{j,k_v,\boldsymbol{\tau}_v}(v) = \sum_{j=0}^{n}\left(\sum_{i=0}^{m} B_{i,k_u,\boldsymbol{\tau}_u}(u)\right) B_{j,k_v,\boldsymbol{\tau}_v}(v)$$
$$= \sum_{j=0}^{n} B_{j,k_v,\boldsymbol{\tau}_v}(v)$$
$$= 1.$$

Surface Evaluation

Consider $\sigma(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{i,j}B_{j,k_v,\boldsymbol{\tau}_v}(v)B_{i,k_u,\boldsymbol{\tau}_u}(u),$

and $\varphi(u,v)=\sum_{j=0}^{n}\sum_{i=0}^{m}P_{i,j}B_{i,k_u,\boldsymbol{\tau}_u}(u)B_{j,k_v,\boldsymbol{\tau}_v}(v).$

Question: What is the difference between $\sigma(u, v)$ and $\varphi(u, v)$?

Partial Derivatives' Evaluation

Consider $\sigma(u,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v).$

How can we compute $\frac{\partial \sigma(u,v)}{\partial u}$ and $\frac{\partial \sigma(u,v)}{\partial v}$?

$$\frac{\partial}{\partial u}\left(\sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v) \right) = \sum_{j=0}^{n} \frac{\partial}{\partial u}\left(\sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) \right) B_{j,k_v,\tau_v}(v).$$

Question: What about $\partial\sigma/\partial v$?

Tensor Product Example 1

Consider curve $C_1(u)$ and $C_2(v)$. Recall that if we seek the minimal distance between $C_1(u)$ and $C_2(v)$, we need to compute the extrema of:

$$\sigma(u,v) = \|C_1(u) - C_2(v)\|^2$$
$$= \langle C_1(u) - C_2(v), C_1(u) - C_2(v)\rangle.$$

To compute the extrema of $\sigma(u,v)$, we differentiate:

$$\frac{d\sigma(u,v)}{du} = 2\langle C_1{}'(u), C_1(u) - C_2(v)\rangle,$$

only to seek the simultaneous zeros of both partials.

$$\frac{d\sigma(u,v)}{du} = 2\langle C_1'(u), C_1(u) - C_2(v)\rangle$$

Tensor Product Example 1 (cont)

Question: How do we represent $C_1'(u)$ and $C_1(u) - C_2(v)$ as tensor product surface (bivariate), like $\sigma(u,v)$?

For $C_1'(u)$:

$$\begin{aligned} C_1'(u) &= \sum P_i \theta_i(u) \\ &= \sum P_i \theta_i(u) \sum \theta_j(v) \\ &= \sum \sum P_i\, \theta_i(u) \theta_j(v). \end{aligned}$$

$$\frac{d\sigma(u,v)}{du} = 2\langle C_1{}'(u), C_1(u) - C_2(v)\rangle$$

Tensor Product Example 1 (cont)

Same for $C_1(u) - C_2(v)$:

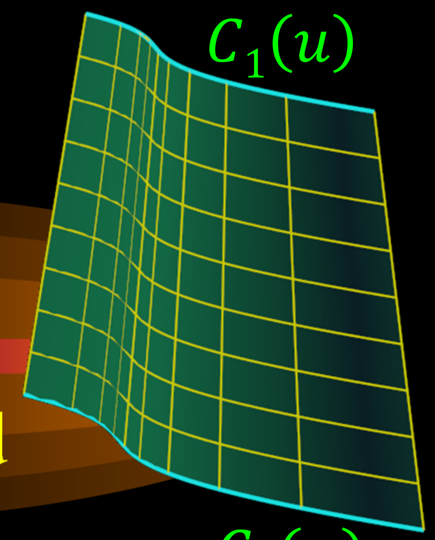
$C_1(u) - C_2(v) =$

$$= \sum P_i \theta_i(u) - \sum Q_j \theta_j(v)$$

$$= \sum P_i \theta_i(u) \sum \theta_j(v) - \sum Q_j \theta_j(v) \sum \theta_i(u)$$

$$= \sum \sum (P_i - Q_j)\, \theta_i(u) \theta_j(v) .$$

Tensor Product Example 2



Consider two curves $C_1(u) = \sum P_i \theta_i(u)$ and $C_2(u) = \sum Q_i \theta_i(u)$, $u \in [0,1]$, and compute the ruled surface between them:

$$\sigma(u,v) = C_1(u)(1-v) + C_2(u)v, \qquad v \in [0,1]$$
$$= \sum P_i \theta_i(u)\,(1-v) + \sum Q_i \theta_i(u)\, v$$
$$= \sum P_i \theta_i(u)\, \theta_{0,1}(v) + \sum Q_i \theta_i(u)\, \theta_{1,1}(v)$$
$$= \sum_{j=0}^{1} \sum R_{ij} \theta_i(u)\, \theta_{j,1}(v),$$

where $R_{ij} = P_i$ for $j = 0$ and $R_{ij} = Q_i$ for $j = 1$.

Tensor Product Example 3

Consider two tensor product surfaces

$S_1(u,v) = \sum\sum P_{ij}\theta_i(u)\theta_j(v)$ and

$S_2(u,v) = \sum\sum Q_{ij}\theta_i(u)\theta_j(v)$,

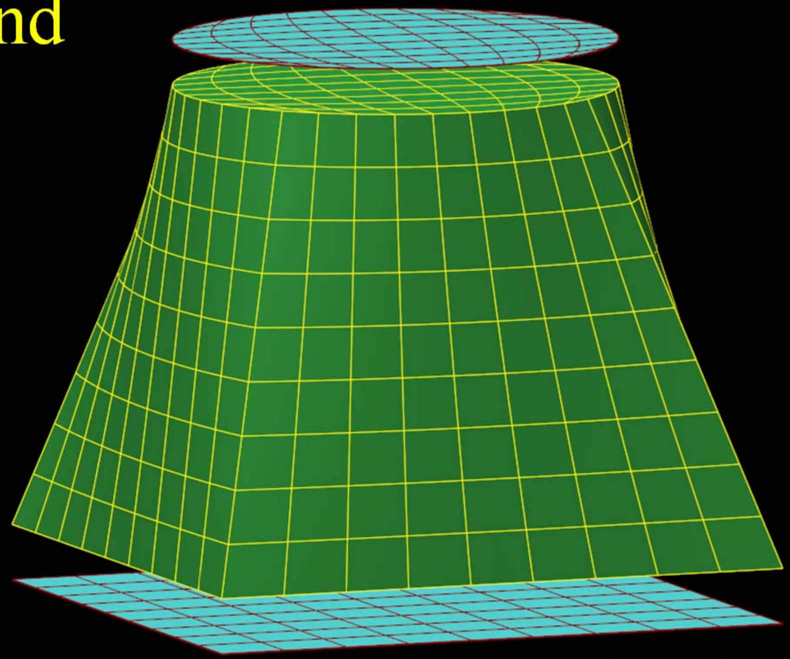
$u, v \in [0,1]$,

and compute the

ruled trivariate between them:

$T(u,v,w) =$

$S_1(u)(1-w) + S_2(u)w, \; w \in [0,1]$



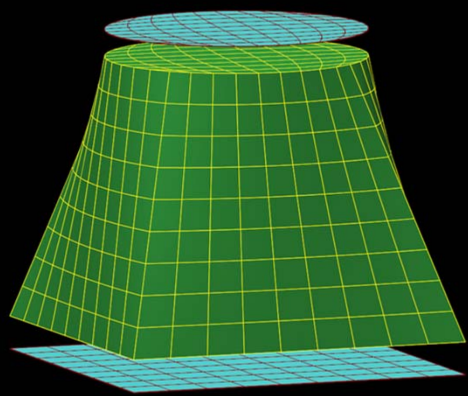
Tensor Product Example 3 (cont.)

$$T(u,v,w) = S_1(u)(1-w) + S_2(u)w, \qquad w \in [0,1]$$

$$= \sum\sum P_{ij}\theta_i(u)\theta_j(v)\,(1-w) + \sum\sum Q_{ij}\theta_i(u)\theta_j(v)\,w$$

$$= \sum\sum P_{ij}\theta_i(u)\theta_j(v)\,\theta_{0,1}(w) + \sum\sum Q_{ij}\theta_i(u)\theta_j(v)\,\theta_{1,1}(w)$$

$$= \sum_{k=0}^{1}\sum\sum R_{ijk}\theta_i(u)\theta_j(v)\theta_{k,1}(w),$$



where $R_{ijk} = P_{ij}$ for $k = 0$ and $R_{ijk} = Q_{ij}$ for $k = 1$.

Tensor Product Example 3 (cont.)

$T(u, v, w)$ represents a solid object. Meaning T represents both the boundary of the geometry and its interior, which, for example, can be heterogeneous.

Very important, for example, for:

- 3D printing of heterogeneous materials:
- Analysis (stress/heat transfer, etc.).

Tensor Product Example 4

- We talked about curves, surfaces, and trivariates.
- Generalizing, we can talk about multivariate functions in the Bezier and B-spline.
- This allows us to represent arbitrary polynomials function in any dimension and range.
- Opens up whole new areas of research,
  - Work in multivariate constraints solving.