Tensor Product Surfaces

**Definition 13.1:**

Consider \( F \) and \( G \), two sets of univariate functions with intervals domains \( U \) and \( V \), respectively,

\[
F = \{ f_i(u) \}_{i=0,m}, \quad G = \{ g_j(v) \}_{j=0,n}.
\]

A surface formed by

\[
h(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} f_i(u) g_j(v)
\]

is called a **tensor product surface** with domain \( U \times V \).

If \( c_{i,j} \in \mathbb{R}^3 \) for all \( i, j \), then \( h \) is a parametric surface.
Example 13.2:

Consider the linear blending functions

\[ F = \{ f_0(u) = 1-u, \ f_1(u) = u \} \quad \text{and} \quad G = \{ g_0(v) = 1-v, \ g_1(v) = v \}, \]

with domain \( U = [0, 1] \) and \( V = [0, 1] \). The tensor product surface

\[
h(u, v) = c_{0,0} f_0(u) g_0(v) + c_{0,1} f_0(u) g_1(v) + c_{1,0} f_1(u) g_0(v) + c_{1,1} f_1(u) g_1(v)
\]

is a bilinear tensor product surface with domain \( U \times V \).

Question: What is the value of \( h(u, v) \) at \( u = 0, 1 \) and \( v = 0, 1 \)?
Definition 13.3:

Consider $P = \{ P_{i,j} \in \mathbb{R}^3 \mid 0 \leq i \leq m, \ 0 \leq j \leq n \}$ and collection of functions $F = \{ \theta_{i,m}(u) \}_{i=0}^m$, $G = \{ \theta_{j,n}(v) \}_{j=0}^n$. The parametric surface

$$\sigma(u, v) = \sum_{i=0}^n \sum_{j=0}^m P_{i,j} \theta_{i,m}(u) \theta_{j,n}(v)$$

is called a degree $m \times n$ tensor product Bezier surface with domain $U \times V = [0, 1] \times [0, 1]$. 

**Question:** What is the value of $\sigma(u, v)$ at $u = 0, 1$ or $v = 0, 1$?
Beziers Surfaces (Cont.)

From properties of Bernstein polynomials

\[
\sigma(u,0) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} \theta_{i,m}(u) \theta_{j,n}(0) = \sum_{i=0}^{m} P_{i,0} \theta_{i,m}(u),
\]

\[
\sigma(u,1) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} \theta_{i,m}(u) \theta_{j,n}(1) = \sum_{i=0}^{m} P_{i,n} \theta_{i,m}(u),
\]

\[
\sigma(0,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} \theta_{i,m}(0) \theta_{j,n}(v) = \sum_{j=0}^{n} P_{0,j} \theta_{j,n}(v),
\]

\[
\sigma(1,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} \theta_{i,m}(1) \theta_{j,n}(v) = \sum_{j=0}^{n} P_{m,j} \theta_{j,n}(v).
\]
A Tensor Product B-spline Surface

Definition 13.4:

Consider \( P = \{ P_{i,j} \in \mathbb{R}^3 \mid 0 \leq i \leq m, \ 0 \leq j \leq n \} \) and collection of functions \( F = \{ B_{i,k_u,\tau_u}(u) \}_{i=0}^{m}, G = \{ B_{j,k_v,\tau_v}(v) \}_{j=0}^{n} \), where \( \tau_u \) and \( \tau_v \) are knot vectors of length \( m + k_u + 2 \) and \( n + k_v + 2 \), respectively.

The parametric surface

\[
\sigma(u, v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v)
\]

is called a degree \( k_u \times k_v \) tensor product B-spline surface with domain

\[
U \times V = [\tau_u(k_u), \tau_u(m+1)] \times [\tau_v(k_v), \tau_v(n+1)].
\]
B-spline Products

Example 13.5: Consider $\tau_u = \tau_v = \{0, 0, 0, 1, 2, 2, 2\}$ and quadratic functions. We draw the different products:

$$B_{0,2}(u)B_{0,2}(v) \quad \text{and} \quad B_{0,2}(u)B_{1,2}(v)$$
Example 13.5 (Cont.):

\[ \tau_u = \tau_v = \{ 0, 0, 0, 1, 2, 2, 2 \}. \]

\[ B_{0,2}(u)B_{2,2}(v) \]

\[ B_{0,2}(u)B_{3,2}(v) \]
Example 13.5 (Cont.):

\[ \tau_u = \tau_v = \{0, 0, 0, 1, 2, 2, 2\}. \]

\[ B_{1,2}(u)B_{1,2}(v) \]

\[ B_{1,2}(u)B_{2,2}(v) \]
Example 13.5 (Cont.):

\[ \tau_u = \tau_v = \{ 0, 0, 0, 1, 2, 2, 2 \}. \]

\[ B_{1,2}(u)B_{3,2}(v) \quad B_{2,2}(u)B_{2,2}(v) \]
Example 13.5 (Cont.):

\[ \tau_u = \tau_v = \{0, 0, 0, 1, 2, 2, 2\}. \]

\[ B_{2,2}(u)B_{3,2}(v) \]

\[ B_{3,2}(u)B_{3,2}(v) \]
Control Mesh

Definition 13.7:

The collection

\[ P = \{ P_{i,j} \in R^3 | 0 \leq i \leq m , \ 0 \leq j \leq n \} \]

is called the control mesh for the Bezier/B-spline surface.
Convex Combination

Lemma 13.8:

Suppose \( \sigma(u, v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v) \) is a tensor product B-spline surface, for the proper domain. Then

\[
1 = \sum_{j=0}^{n} \sum_{i=0}^{m} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v).
\]
Convex Combination (Cont.)

Lemma 13.8 (cont.):

Proof: \[ \sum_{i=0}^{m} B_{i,k_u,\tau_u}(u) = 1, \quad \sum_{j=0}^{n} B_{j,k_v,\tau_v}(v) = 1. \]

Then: \[ \sum_{j=0}^{n} \sum_{i=0}^{m} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v) = \sum_{j=0}^{n} \left( \sum_{i=0}^{m} B_{i,k_u,\tau_u}(u) \right) B_{j,k_v,\tau_v}(v) = \sum_{j=0}^{n} B_{j,k_v,\tau_v}(v) = 1. \]
Surface Evaluation

Consider

\[
\sigma(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{j,k_v,\tau_v}(v) B_{i,k_u,\tau_u}(u),
\]

and

\[
\varphi(u, v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v).
\]

**Question:** What is the difference between \(\sigma(u, v)\) and \(\varphi(u, v)\)?
Partial Derivatives’ Evaluation

Consider \( \sigma(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} B_{i,k_{u},\tau_{u}}(u) B_{j,k_{v},\tau_{v}}(v) \).

How can we compute \( \frac{\partial \sigma(u, v)}{\partial u} \) and \( \frac{\partial \sigma(u, v)}{\partial v} \)?

\[ \frac{\partial}{\partial u} \left( \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} B_{i,k_{u},\tau_{u}}(u) B_{j,k_{v},\tau_{v}}(v) \right) = \sum_{i=0}^{n} \frac{\partial}{\partial u} \left( \sum_{j=0}^{m} P_{i,j} B_{i,k_{u},\tau_{u}}(u) \right) B_{j,k_{v},\tau_{v}}(v). \]

**Question:** What about \( \partial \sigma/\partial v \)?
Tensor Product Example 1

Consider curve $C_1(u)$ and $C_2(v)$. Recall that if we seek the minimal distance between $C_1(u)$ and $C_2(v)$, we need to compute the extrema of:

$$\sigma(u, v) = \|C_1(u) - C_2(v)\|^2$$

$$= \langle C_1(u) - C_2(v), C_1(u) - C_2(v) \rangle.$$

To compute the extrema of $\sigma(u, v)$, we differentiate:

$$\frac{d\sigma(u,v)}{du} = 2\langle C_1'(u), C_1(u) - C_2(v) \rangle,$$

only to seek the simultaneous zeros of both partials.
Question: How do we represent $C_1'(u)$ and $C_1(u) - C_2(v)$ as tensor product surface (bivariate), like $\sigma(u, v)$?

For $C_1'(u)$:

$$C_1'(u) = \sum P_i \theta_i(u)$$

$$= \sum P_i \theta_i(u) \sum \theta_j(v)$$

$$= \sum \sum P_i \theta_i(u) \theta_j(v).$$
\[ \frac{d\sigma(u,v)}{du} = 2\langle C'_1(u), C_1(u) - C_2(v) \rangle \]

Tensor Product Example 1 (cont)

Same for \( C_1(u) - C_2(v) \):

\[ C_1(u) - C_2(v) = \]

\[ = \sum P_i \theta_i(u) - \sum Q_j \theta_j(v) \]

\[ = \sum P_i \theta_i(u) \sum \theta_j(v) - \sum Q_j \theta_j(v) \sum \theta_i(u) \]

\[ = \sum \sum (P_i - Q_j) \theta_i(u) \theta_j(v). \]
Tensor Product Example 2

Consider two curves $C_1(u) = \sum P_i \theta_i(u)$ and $C_2(u) = \sum Q_i \theta_i(u)$, $u \in [0,1]$, and compute the ruled surface between them:

$$\sigma(u, v) = C_1(u)(1 - v) + C_2(u)v, \quad v \in [0,1]$$

$$= \sum P_i \theta_i(u) (1 - v) + \sum Q_i \theta_i(u) v$$

$$= \sum P_i \theta_i(u) \theta_{0,1}(v) + \sum Q_i \theta_i(u) \theta_{1,1}(v)$$

$$= \sum_{j=0}^{1} \sum R_{ij} \theta_i(u) \theta_{j,1}(v),$$

where $R_{ij} = P_i$ for $j = 0$ and $R_{ij} = Q_i$ for $j = 1$. 
Tensor Product Example 3

Consider two tensor product surfaces

\[ S_1(u, v) = \sum \sum P_{ij} \theta_i(u) \theta_j(v) \] and
\[ S_2(u, v) = \sum \sum Q_{ij} \theta_i(u) \theta_j(v), \]
\[ u, v \in [0,1], \]

and compute the

ruled trivariate between them:

\[ T(u, v, w) = \]
\[ S_1(u)(1 - w) + S_2(u)w, \quad w \in [0,1] \]
Tensor Product Example 3 (cont.)

\[ T(u, v, w) = S_1(u)(1 - w) + S_2(u)w, \quad w \in [0,1] \]
\[ = \sum \sum P_{ij} \theta_i(u) \theta_j(v) (1 - w) + \]
\[ \sum \sum Q_{ij} \theta_i(u) \theta_j(v) w \]
\[ = \sum \sum P_{ij} \theta_i(u) \theta_j(v) \theta_{0,1}(w) + \]
\[ \sum \sum Q_{ij} \theta_i(u) \theta_j(v) \theta_{1,1}(w) \]
\[ = \sum_{k=0}^{1} \sum \sum R_{ijk} \theta_i(u) \theta_j(v) \theta_{k,1}(w), \]

where \( R_{ijk} = P_{ij} \) for \( k = 0 \) and \( R_{ijk} = Q_{ij} \) for \( k = 1 \).
Tensor Product Example 3 (cont.)

\( T(u, v, w) \) represents a solid object. Meaning \( T \) represents both the boundary of the geometry and its interior, which, for example, can be heterogeneous.

Very important, for example, for:

- 3D printing of heterogeneous materials:
- Analysis (stress/heat transfer, etc.).
Tensor Product Example 4

- We talked about curves, surfaces, and trivariates.
- Generalizing, we can talk about multivariate functions in the Bezier and B-spline.
- This allows us to represent arbitrary polynomials function in any dimension and range.
- Opens up whole new areas of research,
  - Work in multivariate constraints solving.