Computer Aided Geometric Design

Representations of FreeForms

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based on a book by Cohen, Riesenfeld, & Elber

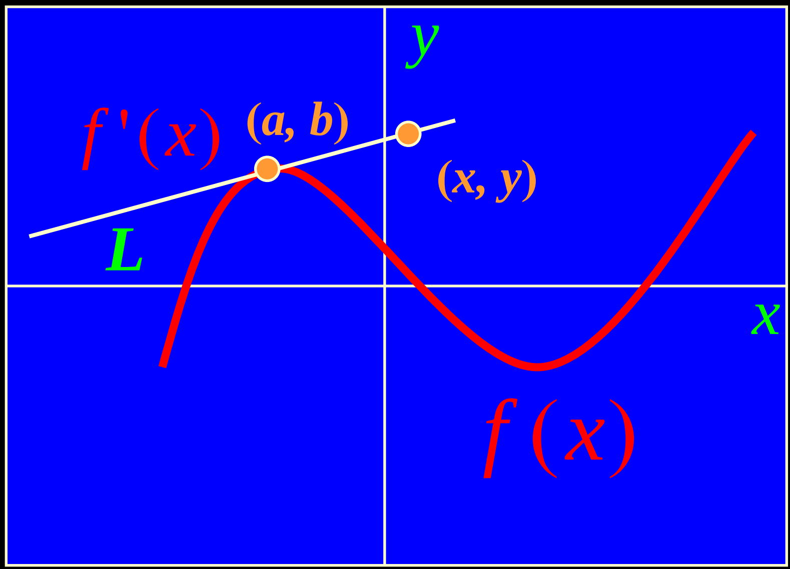
Definition 2.1

A curve formulation is called an **explicit function** if for a given formula of one variable $y = f(x)$, the set of ordered pairs $\{(x, f(x))\}$, for all x in the domain, is a complete description of the curve, and for each first element of the ordered pair, x, there is a unique second element, $f(x)$.

Tangent Vector

At point (a, b), where $b = f(a)$, the **tangent line** is given by

$$L: (y - b) / (x - a) = f`(a).$$



Question: To what is the **normal** to the curve equal?

Example 2.2

Question: Given the polynomial $f(x) = \Sigma_{i=0}^{m} a_i x^i$, how many points of the form $\{(x_j, y_j)\}_{j=0}^{n}$ can $f(x)$ **interpolate**?

Question: What if $n > m$?

Question: Is **interpolation** rigid motion invariant?

Example 2.2 (Cont.)

Consider n data points, $\{(x_j, y_j)\}_{j=0}^{n}$, and a curve that must represent the data, $f(x)$, with only m degrees of freedom, $m < n$. The **least squares** form of approximation minimizes the error E, defined as

$$E = \sum_{j=0}^{n} \left| f(x_j) - y_j \right|^2 .$$

Question: What is the meaning of this distance?

Question: Is **least squares** rigid motion invariant?

Example 2.2 (Cont.)

Consider the quadratic function $f(x) = a + bx + cx^2$. Then,

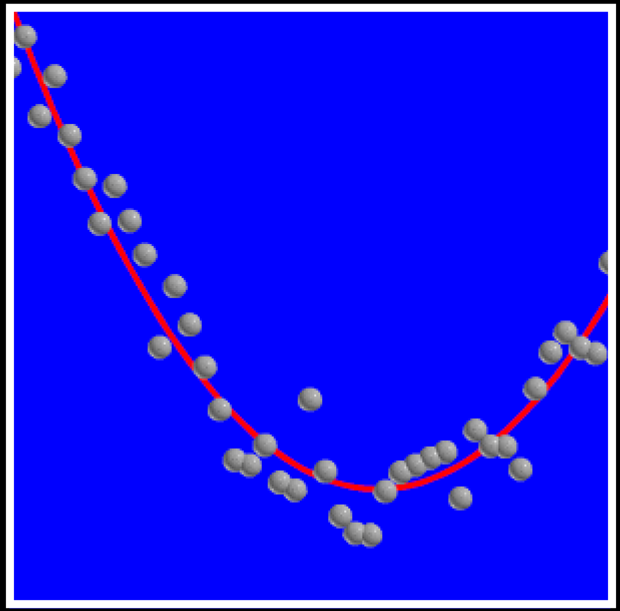
$$E = \sum_{j=0}^{n} \left| a + bx_j + cx_j^2 - y_j \right|^2 = \sum_{j=0}^{n} \left(a + bx_j + cx_j^2 - y_j\right)\left(a + bx_j + cx_j^2 - y_j\right).$$

or (similarly for $\partial E/\partial b$, $\partial E/\partial c$),

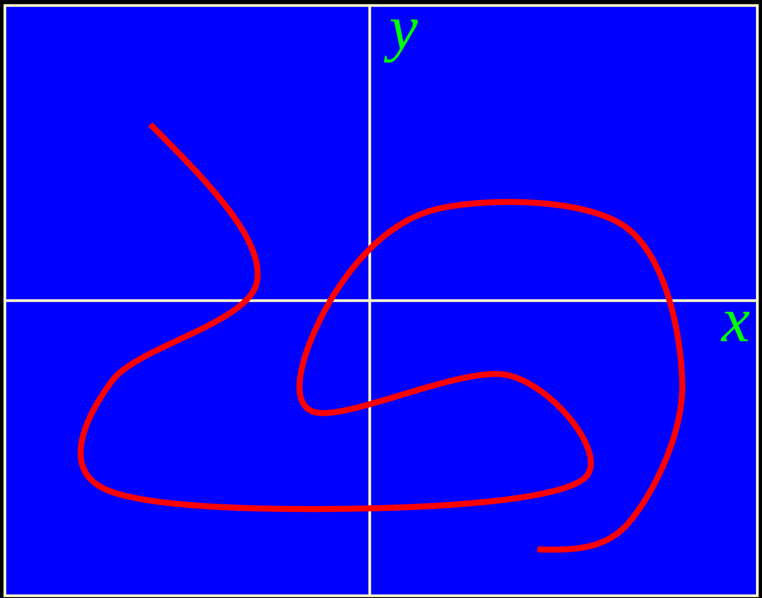
$$\frac{\partial E}{\partial a} = \sum_{j=0}^{n} 2\left(a + bx_j + cx_j^2 - y_j\right) = 0.$$

forming three linear equations

to solve for (a, b, c).



Question: Can the following curve be represented as an explicit curve?

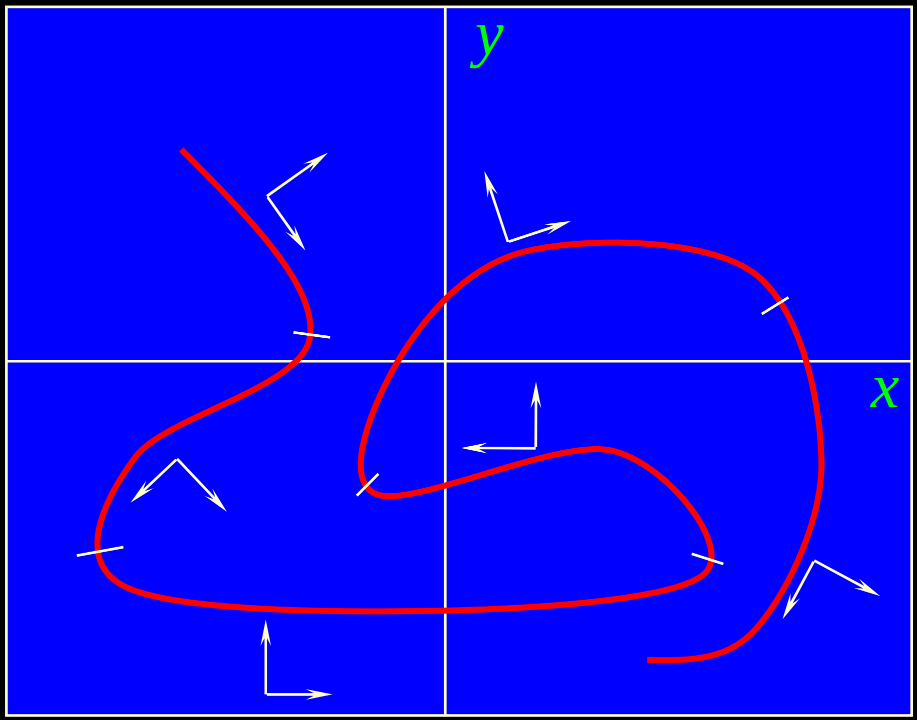


One can parameterize this curve as a

piecewise

explicit

curve:



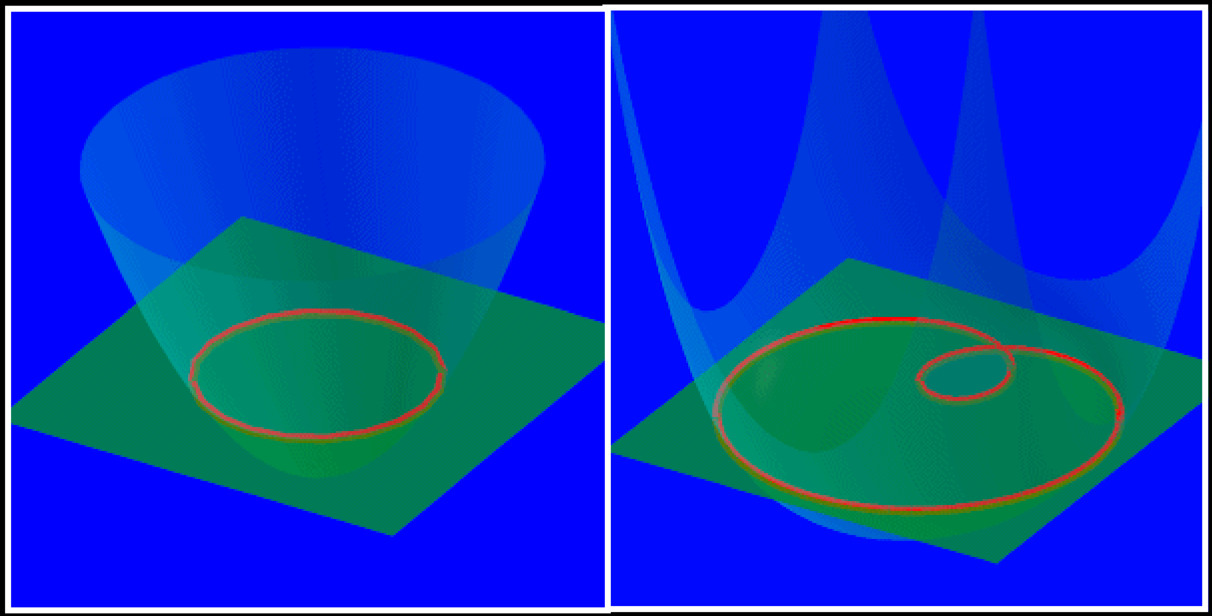
Definition 2.3

The **implicit curve (surface)** is the solution (zero) set to an equation of the form $f(x, y)=0$ for curves (or $f(x, y, z)=0$ for surfaces).

$x^2+y^2-1=0,$

and

$(x^2+y^2+2ay)^2-a^2(x^2+y^2) = 0$



Definition 2.4

An implicit bivariate polynomial of degree n

has the form

$$0 = \sum_{i=0}^{n} \sum_{j=0}^{j=i} a_{i-j,j} x^{i-j} y^{j}.$$

Question: How many terms a degree n implicit has?

Quadratic Implicit Equations

The quadratic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

can also be written as

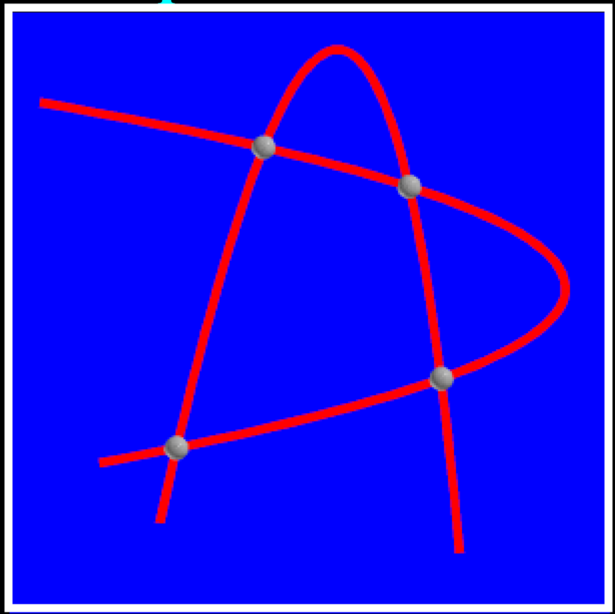
$$\begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ D & B & 0 \\ F & E & C \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix} = 0$$ or as $$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0.$$

This quadratic form contains ellipses, hyperbolas, and parabolas, but also circles and lines as special cases.

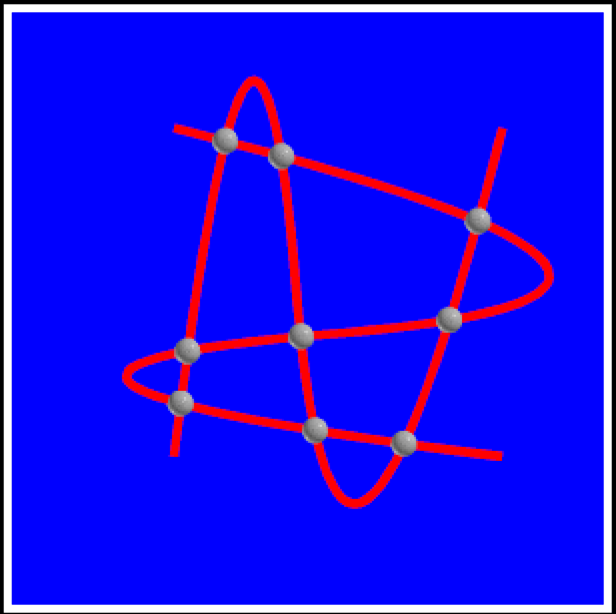
Theorem 2.5 (Bezout's Theorem)

A curve of n th degree and a curve of m th degree intersect in at most $n\,m$ points.

Two **quadratic** curves



Two **cubic** curves



Differentiation of Implicit Functions

Assume a small piece of the implicit function can be represented as a function of x, $f(x, y(x)) = 0$.

Differentiating,

$$0 = \frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad or \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

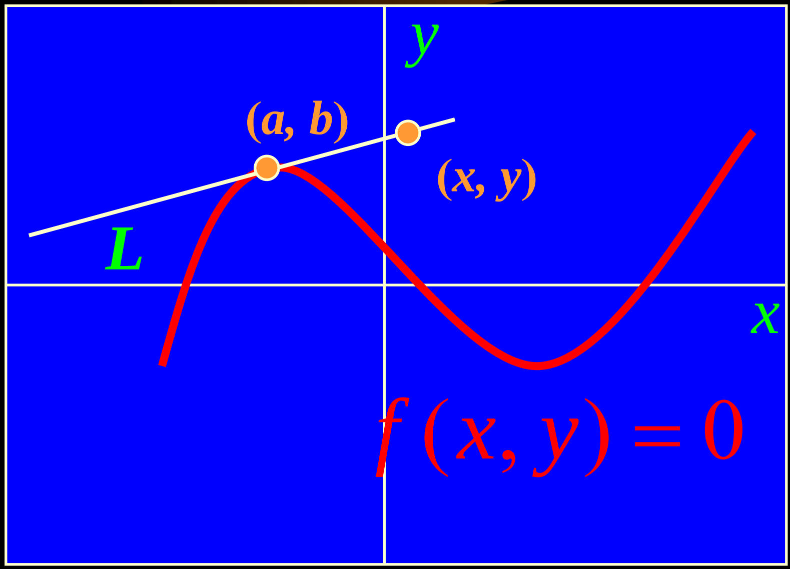
Question: Given $f(x, y)$ what is

$$grad \quad f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)?$$

Tangent Line

Consider location (a, b) and the (tangent) line equation of

$$\frac{y-b}{x-a} = -\frac{\frac{\partial f}{\partial x}(a,b)}{\frac{\partial f}{\partial y}(a,b)}$$



or $\frac{\partial f}{\partial y}(a,b)(y-b) + \frac{\partial f}{\partial x}(a,b)(x-a) = 0.$

Example

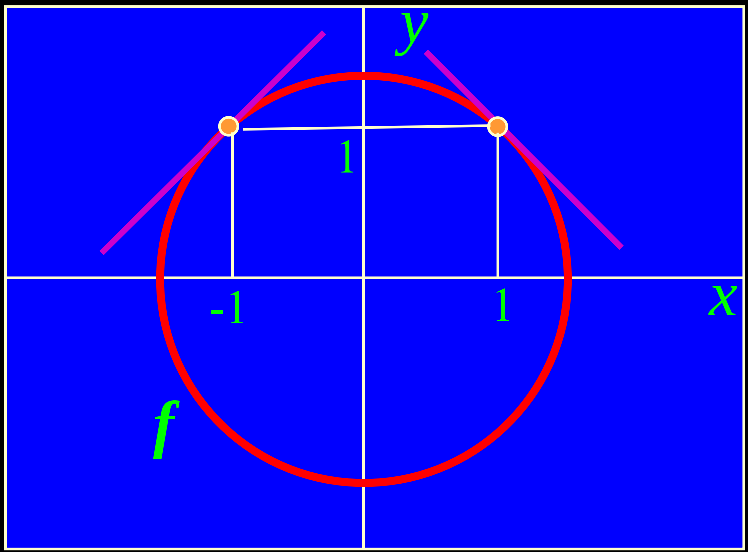
Consider the implicit function of a circle: $f : x^2 + y^2 = 1$.

Assume a parameterization $f : x^2 + y^2(x) = 1$.

Then,

$\partial f / \partial x = 2x + 2y(x)\, y'(x) = 0$

or $y' = -x / y$.



Example

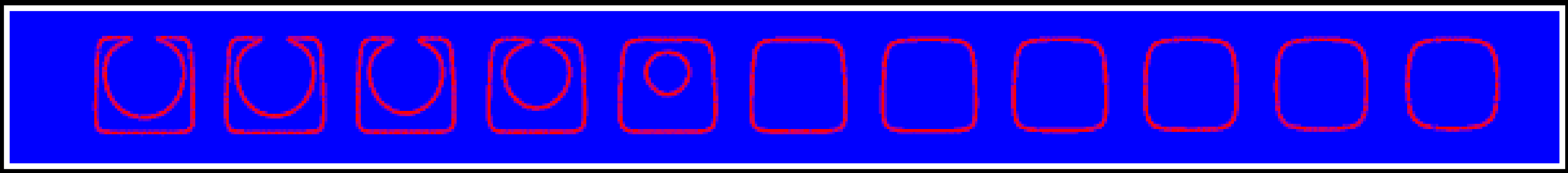
Question: Given implicit forms

$$f_1(x, y) = 0 \text{ and } f_2(x, y) = 0,$$

to what is the form

$$f(x, y) = \alpha f_1(x, y) + (1 - \alpha) f_2(x, y) = 0$$

equal, for $0 \leq \alpha \leq 1$? What if $\alpha < 0$? $\alpha > 1$?



Drawing Implicit Functions

Question: How can we draw **explicit functions**?

Question: How can we draw **implicit functions**?

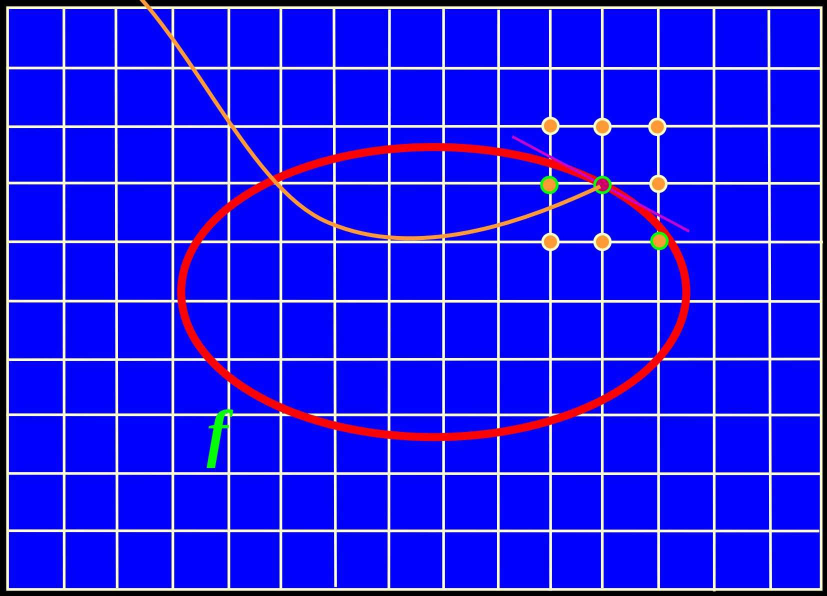
The **tangent** to the **implicit form** at (a, b) equals:

$$\left(-\frac{\partial f}{\partial y}(a,b), \frac{\partial f}{\partial x}(a,b)\right) \quad \text{or} \quad \left(\frac{\partial f}{\partial y}(a,b), -\frac{\partial f}{\partial x}(a,b)\right).$$

Drawing Implicit Functions (Cont.)

The first approach to graphing or drawing is by **marching** in the **tangent direction** on the fixed grid.

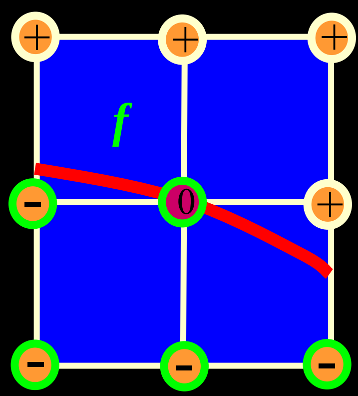
From position (x_0, y_0) one has **eight** possible options based on the signs of $\partial f / \partial x$ and $\partial f / \partial y$ (the **gradient**).

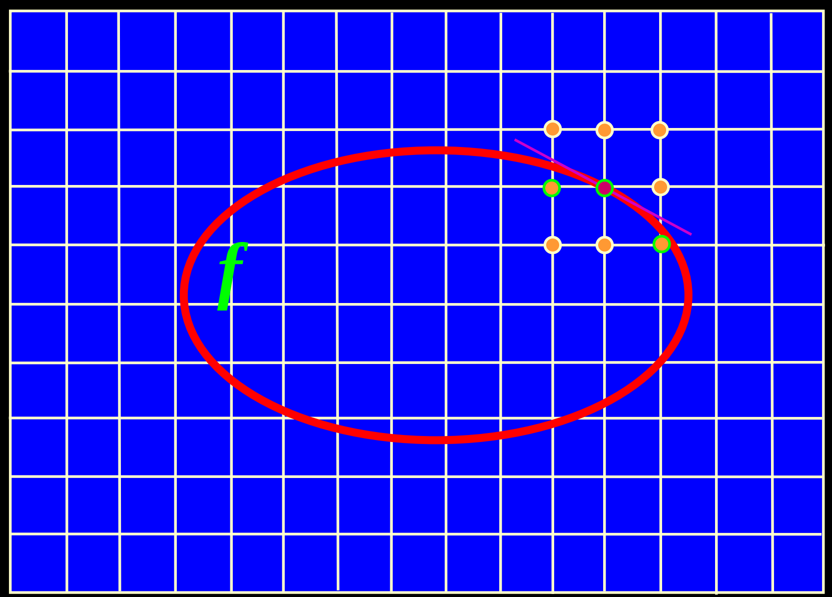


Question: What is the **sign** of the function at the grid points?

Drawing Implicit Functions (Cont.)

The second approach to drawing implicits is by **sampling the function at the grid points** followed by the **examination of *f* 's signs at the four corners of each cell:**



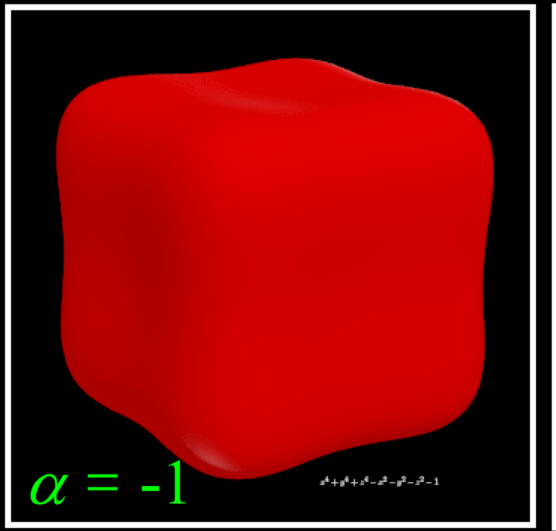


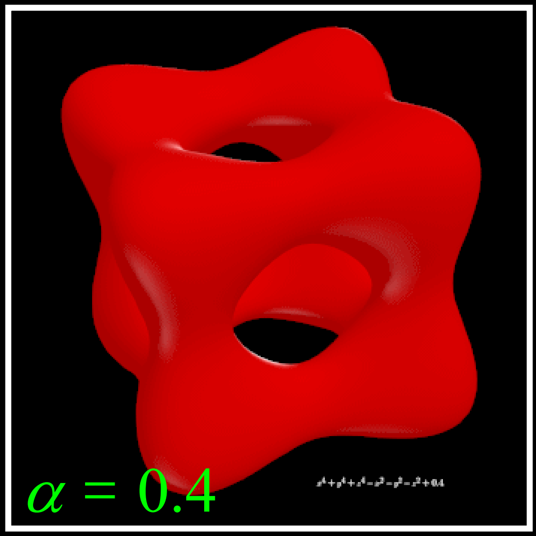
Drawing Implicit Functions (Cont.)

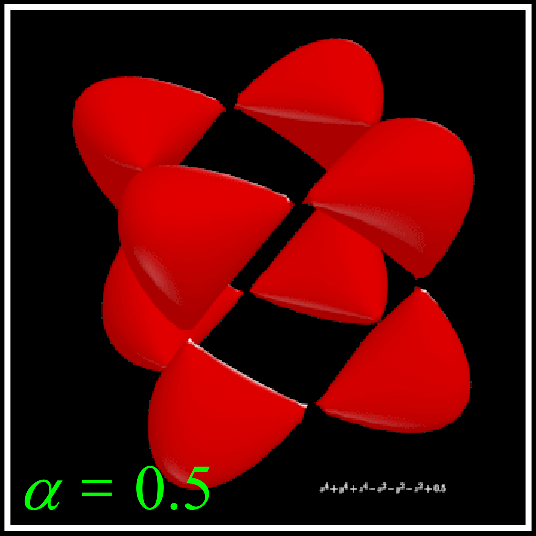
This second approach is also known as **Marching Cubes**, since in one higher dimension, these cells are, in fact, **cubes**.

The following figures are examples of the use of **Marching Cubes**:

Cuboid: $x^4 + y^4 + z^4 - x^2 - y^2 - z^2 + \alpha = 0$



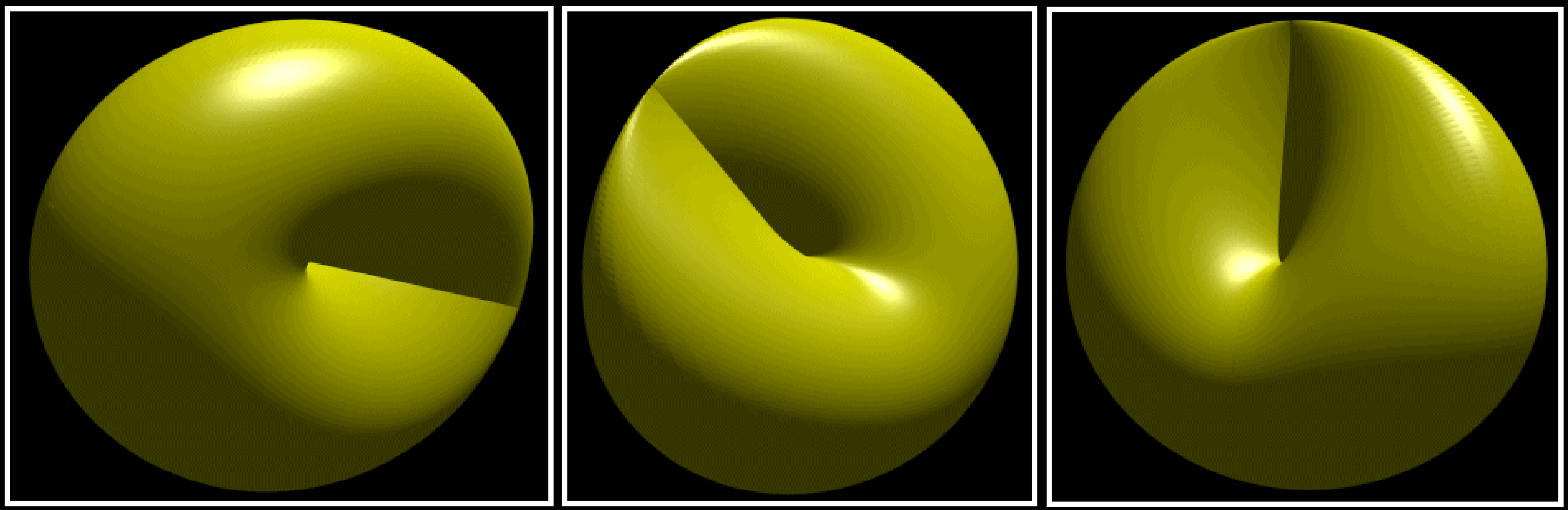




Drawing Implicit Functions (Cont.)

Principal Curvatures' Surface

$$(k_1 x^2 + k_2 y^2)(x^2 + y^2 + z^2) - 2z(x^2 + y^2) = 0$$



Drawing Implicit Functions (Cont.)

Steiner (or Roman) surface:

$$xy + xz + yz - xyz = 0$$

Definition 2.6

In the **parametric representation** of a curve or surface, each coordinate is given as a function of one or two parameters, i.e.,

$C(t) = (x(t), y(t), z(t))$ represents a curve and,

$S(u, v) = (x(u, v), y(u, v), z(u ,v))$ represents a surface.

Definition 2.12

The **arc length parameterization** of a curve is a parameterization that assigns the **position of the curve** as a function of the **length of the curve** measured from a specific point.

Example 2.13

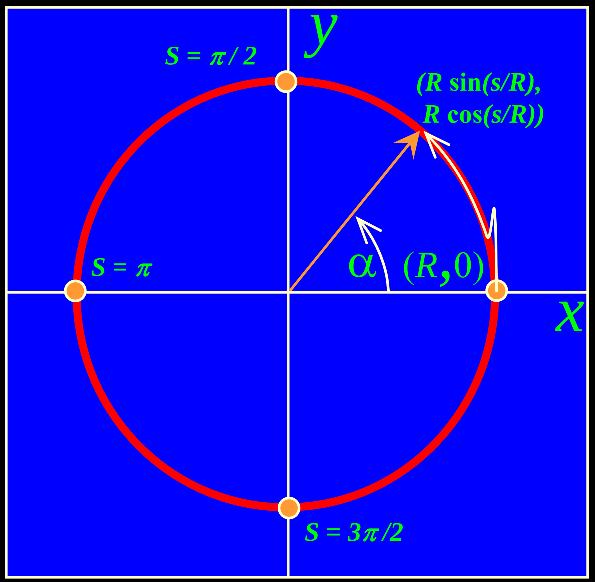
The arc length parameterization of a circle starts from $(R, 0)$. If α is the angle, measured counter-clockwise,

$x = R \sin \alpha = R \sin(s/R)$ and

$y = R \cos \alpha = R \cos(s/R)$,

where s measures the arc length.

For a circle, the arc length is the length of its circumference between two specific points.



Example 2.15

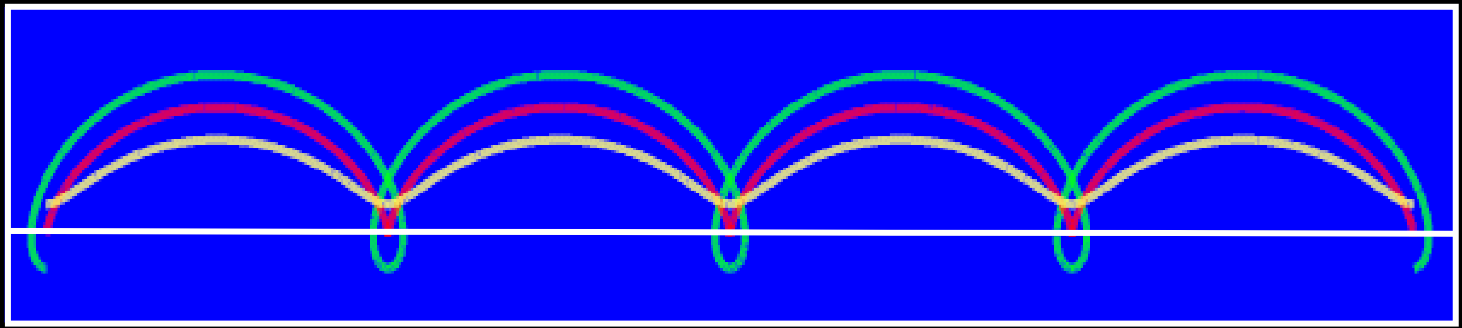
A cycloid is a curve that marks the path of a fixed point on the circumference of a wheel of radius R during the rotation of the wheel. This can be represented by:

$$f(\phi): \quad x(\phi) = R(\phi - \alpha \sin \phi), \quad y(\phi) = R(1 - \alpha \cos \phi)$$

This form can be turned into the explicit formulation of

$$x = R\cos^{-1}\left(\frac{R-y}{\alpha R}\right) - \sqrt{\alpha^2 R^2 - (y-R)^2} \text{ as } \phi = \cos^{-1}((R - y)/(\alpha R)).$$

$\alpha = 1$
$\alpha = 0.5$
$\alpha = 1.5$

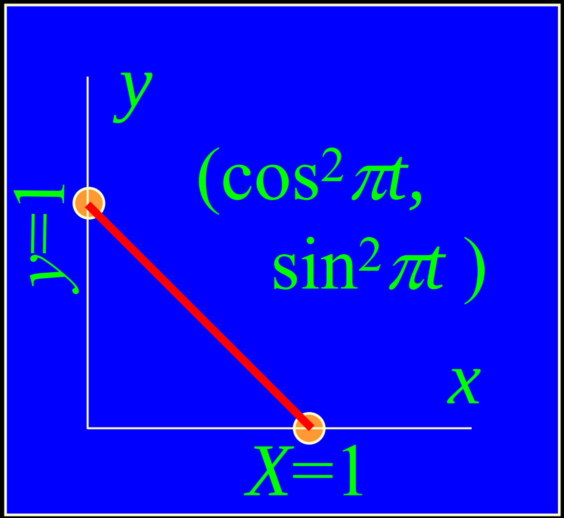


Example 2.17

Consider the curve $(\cos^2 \pi t, \sin^2 \pi t)$ that clearly looks like a trigonometric curve.

Using the trigonometric identity, $\cos^2 a + \sin^2 a = 1$ for all real values of a, one finds the curve identical to the line

$x+y = 1$, where $0 \leq x$ and $0 \leq y$.



Example 2.18

Interpolate three points by a quadratic polynomial. Data equals (0,0), (1,1), (3,0), in that order. Select the following three different parameterizations (selection of t value at Data):

Data	explicit	uniform	chord-len
(0, 0)	0	0	0
(1, 1)	1	1	$\sqrt{2}$
(3, 0)	3	2	$\sqrt{2} + \sqrt{5}$

Question: Would the three quadratic parametric forms be identical?

Example 2.18 (Cont.)

The quadratic parametric form equals

$$p_x(t) = a_x t^2 + b_x t + c_x \qquad\qquad p_y(t) = a_y t^2 + b_y t + c_y$$

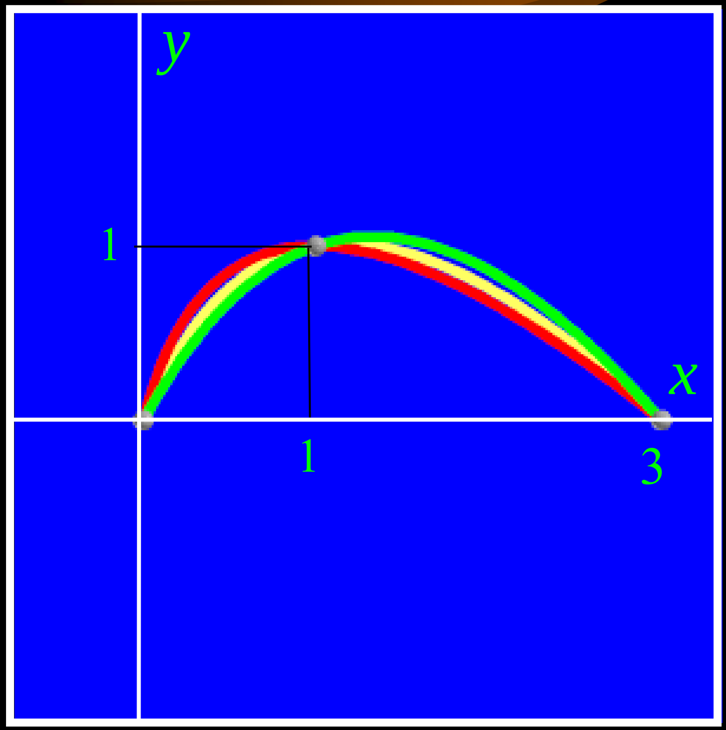
For the **explicit** case, $p_x(t) = t$ and for $p_y(t)$, we end up with the following set of linear equations:

$$\begin{cases} p_y(0) = 0 = a_y 0^2 + b_y 0 + c_y, & t = 0 \\ p_y(1) = 1 = a_y 1^2 + b_y 1 + c_y, & t = 1 \\ p_y(3) = 0 = a_y 3^2 + b_y 3 + c_y, & t = 3 \end{cases}$$

or $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a_y \\ b_y \\ c_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $p_y(t) = -1/2\, t^2 + 3/2\, t$.

Example 2.18 (Cont.)

Similar systems of equations can be set for the uniform and chord-length cases. The end result is differently looking quadratic parametric forms that interpolate the three given Data points.



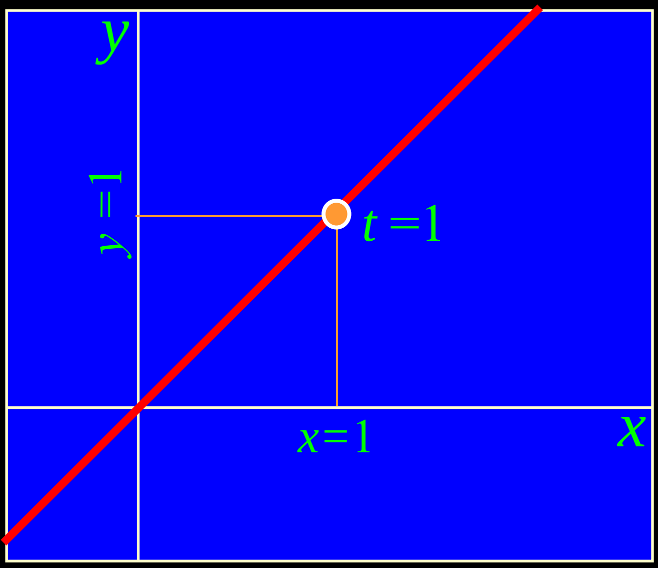
Uniform, Chord Length, Centripedal

Geometric vs. Parametric Continuity

Question: What does the graph of

$$\left(x(t), y(t)\right) = \begin{cases} (t, t), & \text{for } t < 1 \\ (t^2, t^2), & \text{for } t \geq 1. \end{cases}$$

looks like?



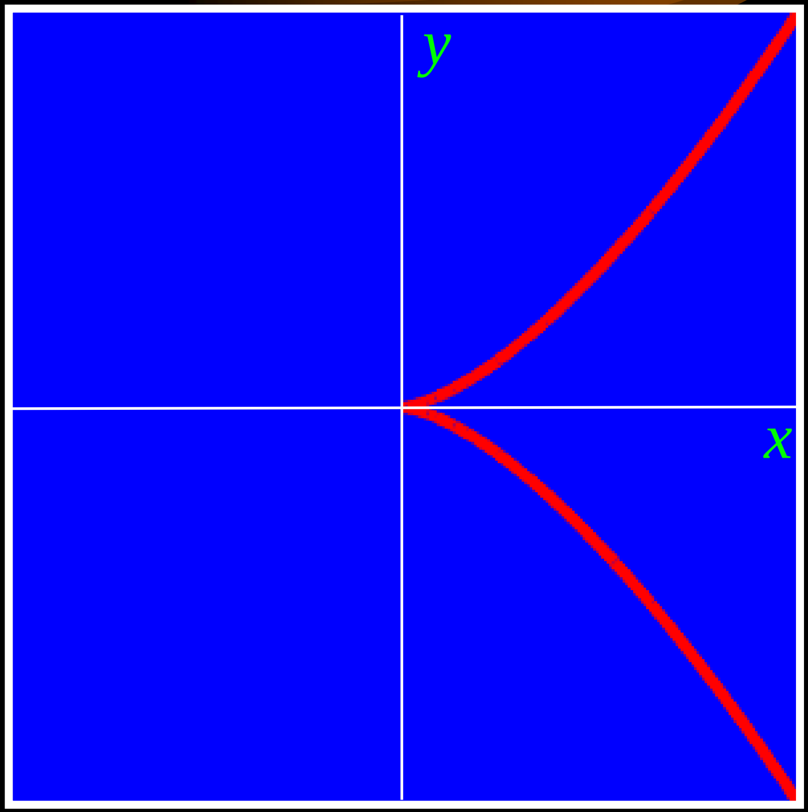
Question: What does the graph of (t, t^2) look like?

Geometric vs. Parametric Continuity

Question: What does the graph of

$$(t^2, t^3)$$

look like?



Conversions