Computer Aided Geometric Design

Differential Geometry for Space Curves

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based on a book by Cohen, Riesenfeld, & Elber

Definition 4.1

$\gamma : I_1$ into $\boldsymbol{R}^3$, $p : I_2$ into/onto I_1, γ and p are **composable functions;** $\gamma(p(u)) : I_2$ into $\boldsymbol{R}^3$ is a **reparametrization** of $\gamma(t)$. This is also called a **change of parameter** from t to u.

Both $\gamma(t)$ and $\gamma(p(u))$ have the same graph.

Definition 4.2

$\gamma(t)$ is a **regular parametric representation** if

for all $t \in I$, the

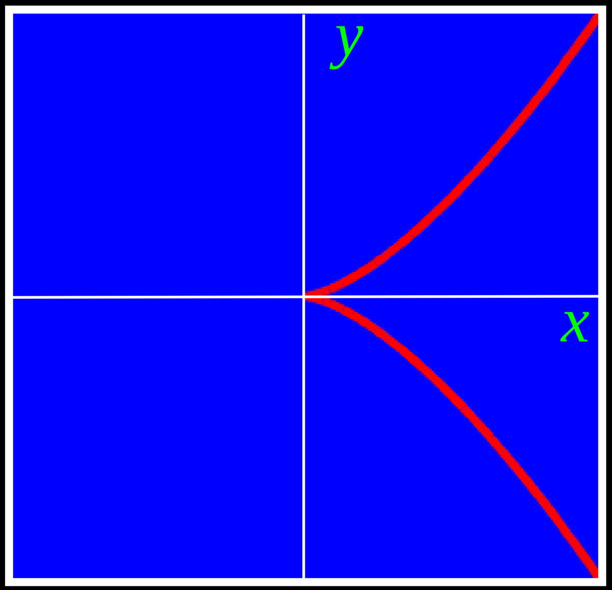
following holds:

$$1.\ \gamma(t) \in C^{(1)},$$

and

$$2.\ \gamma'(t) \neq 0.$$

Question: Why is (t^2, t^3) irregular?



Definition 4.3

$p = p(u)$ is called an

allowable change of parameter

if, for all $u \in I_2$, $p(u) \in C^{(1)}$ and $p'(u) \neq 0$.

Lemma 4.4

The result of reparametrizing a regular parametric representation γ using an allowable change of parameter $p(u)$ is another regular parametric representation $\gamma(p(u))$ representing the same curve or, a part of the same curve.

Proof

Let $p(u)$ be a regular change of parameter such that $\beta(u) = \gamma(p(u))$. Then, $\beta'(u) = \gamma'(p(u))\, p'(u)$.

Definition 4.6

A regular representation $\gamma(t)$, $t \in I_1$, is related to another regular representation $\phi(u)$, $u \in I_2$, if there exists an allowable change of parameter $t = p(u)$ such that:

i) $p(I_2) = I_1$,

ii) $\gamma(p(u)) = \phi(u)$.

Lemma 4.7

The relation in Definition 4.6 is an equivalent relation for all regular parametric representations.

Recall that the set of elements R is called an equivalent relation, $\sim$, if for every element $x, y, z \in R$,

- $x \sim x$ (reflexivity)
- $x \sim y$ implies $y \sim x$ (commutativity)
- $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity)

Definition 4.8

Each equivalence class of the relation,

an allowable change of parameter,

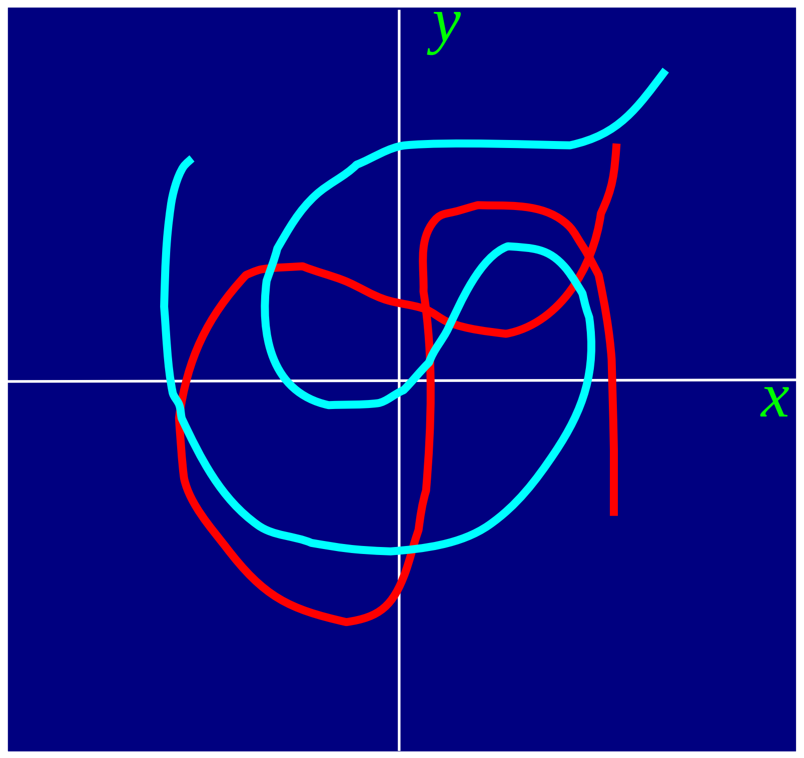
defined in **Definition 4.3,** is called a

regular curve.

Definition 8

A **simple curve** is a curve that does not cross itself.

More analytically, $\gamma(t)$ is a simple curve if it is an injection, i.e., for all $t_1, t_2 \in I, \gamma(t_1) \neq \gamma(t_2)$.



$\pi = 2$?

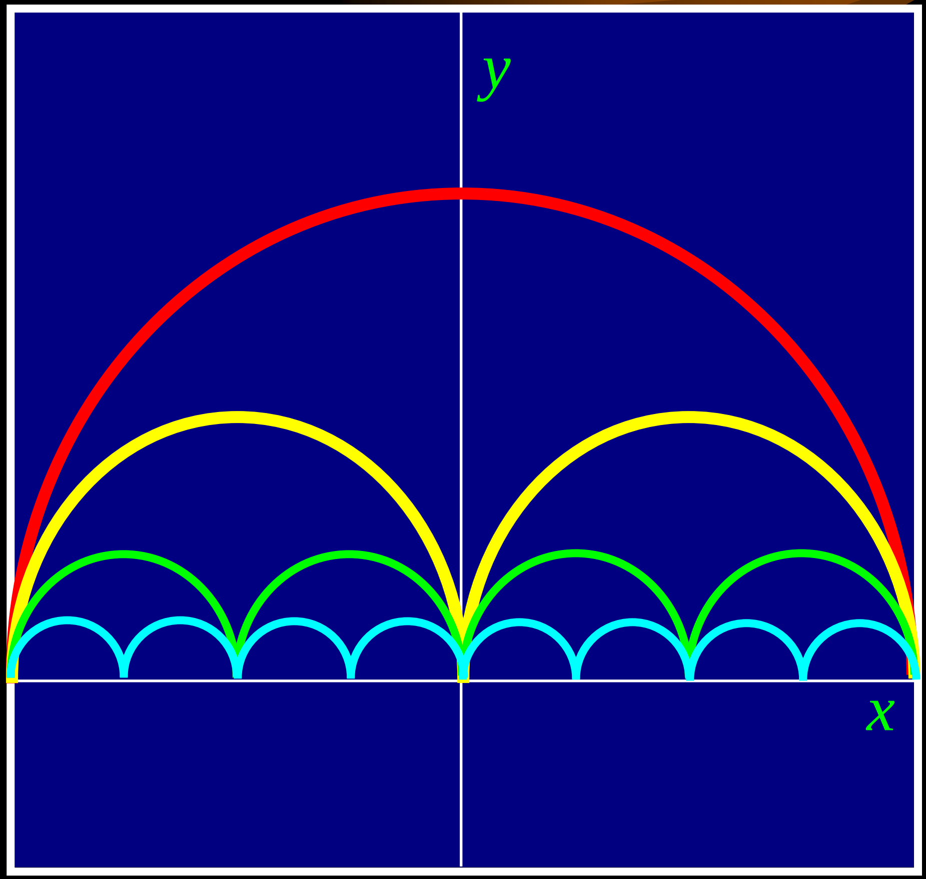
Assume an arc of radius $r = 1$.

Length $= \pi r = \pi$

Length $= 2\pi(r/2) = \pi$

Length $= 4\pi(r/4) = \pi$

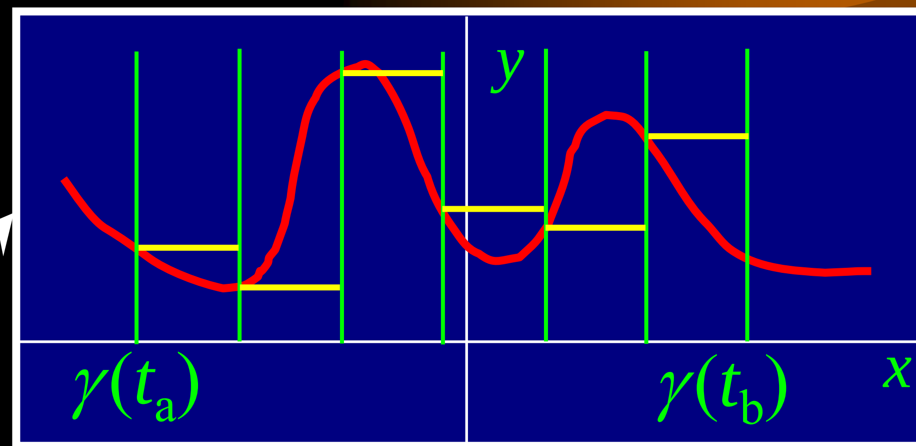
Length $= 8\pi(r/8) = \pi$



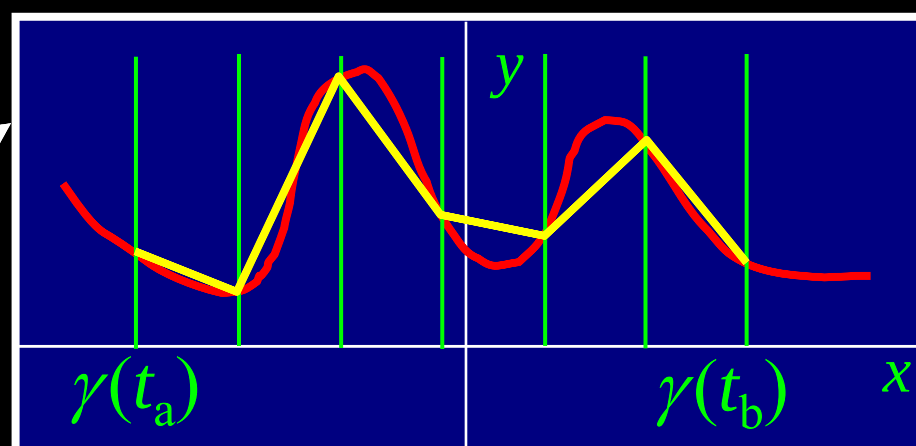
Arc Length Parameterization

Given a arbitrary regular curve, we seek the length of the curve between two distinct parameters, $\gamma(t_a)$ and $\gamma(t_b)$.

Piecewise constant approximation.

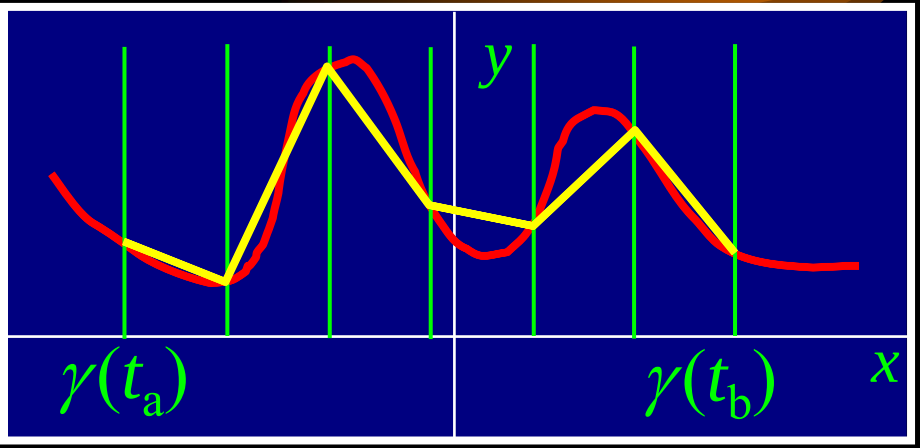


Piecewise linear approximation.



Arc Length Parameterization

Let s be the arc length. Then, examine the piece-wise linear approximation:



$$s \approx \sum_{i=0}^{n-1} \left\| \gamma(t_{i+1}) - \gamma(t_i) \right\|$$

$$= \sum_{i=0}^{n-1} \left\| \frac{\gamma(t_{i+1}) - \gamma(t_i)}{t_{i+1} - t_i} \right\| (t_{i+1} - t_i),$$

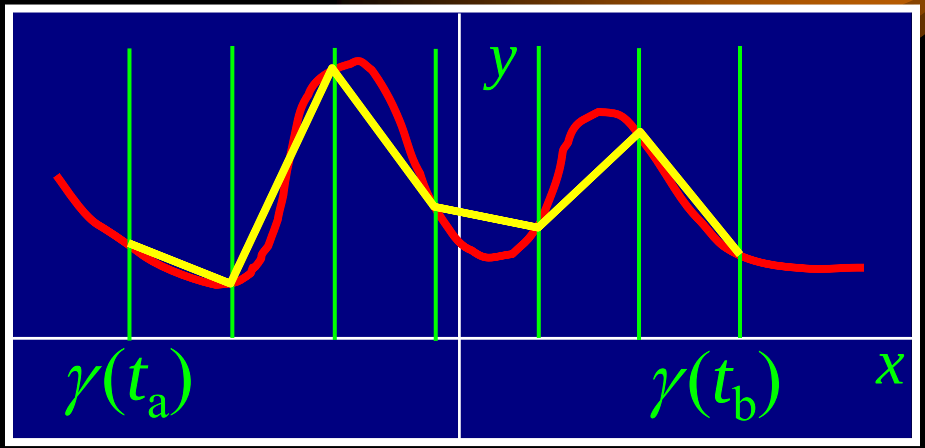
where

$$\left\| \gamma(t_{i+1}) - \gamma(t_i) \right\| = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}.$$

Arc Length Parameterization

On the limit:

$$s = \int_{t_0}^{t} \|\gamma'(\tau)\| d\tau,$$



and from the fundamental theorem of calculus,

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\gamma'(\tau)\| d\tau = \|\gamma'(t)\|.$$

Question: What is the meaning of $\|\gamma'(t)\|$?

Arc Length Parameterization

Assume $\gamma(s)$ is parameterized by arc length. Then $ds/ds = 1 = \|\gamma'(s)\|$. In contrast, assume $\gamma(t)$ is a unit speed curve, or $\|\gamma'(t)\| = 1$, for all t. Then,

$$s = \int_{t=0}^{t} \|\gamma'(t)\| dt = t.$$

Corollary: γ has an arc length parameterization if and only if $\|\gamma'(t)\| = 1$.

Arc Length Parameterization

Arc length or uniform speed curves show up in numerous cases:

- A car, train or an airplane traveling at a constant speed on the road/railway/flight path.
- A numerically controlled NC machine cutting metal at a constant speed.

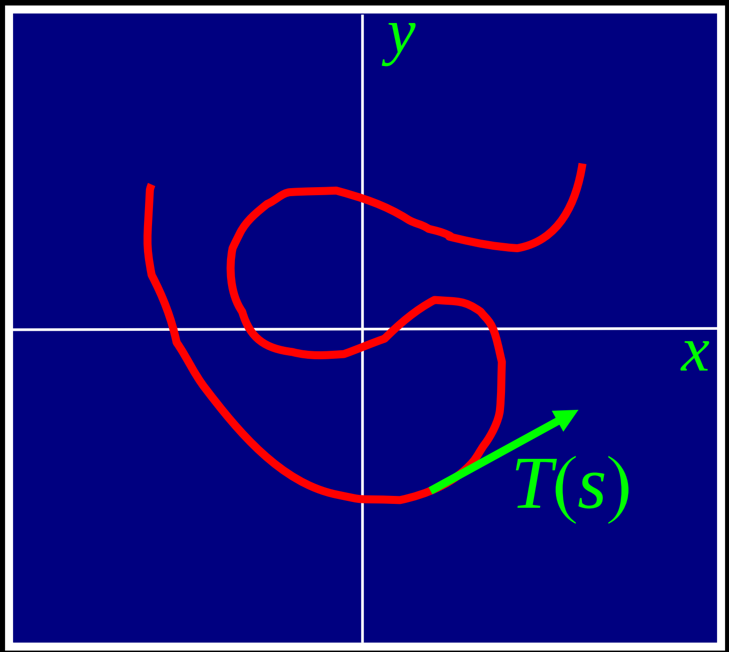
Question: Given a polynomial or rational parametric function $\gamma(t)$, how difficult is it to find $\gamma(s)$, s arc length?

Intrinsic Vectors of Space Curves

Until otherwise stated, we assume $\gamma(s)$, or arc length parameterization, s, and $\|\gamma'(s)\| = 1$.

Definition 4.12

$T = T(s) = \gamma'(s)$ is called the **unit tangent** to the curve γ since $\|T(s)\| = 1$ for all s.

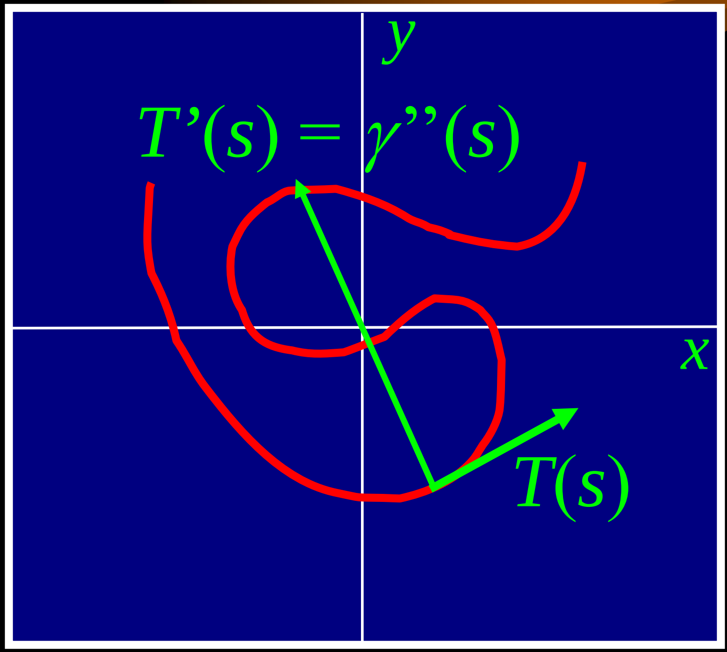


Intrinsic Vectors of Space Curves

Definition 4.13

$T' = T'(s) = \gamma''(s)$ is called

the **curvature vector**.

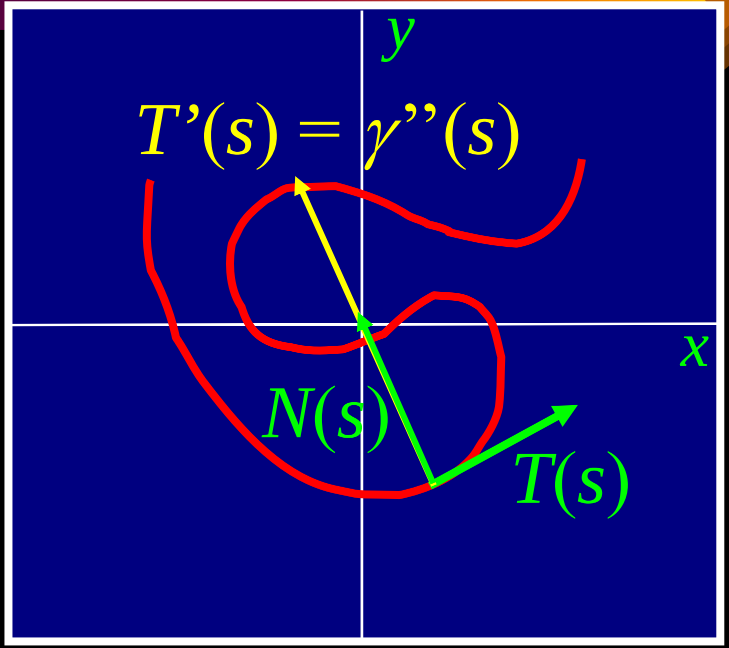


Question: What is the direction of $T'(s)$?

Intrinsic Vectors of Space Curves

Definition 4.14

$\kappa = \kappa(s) = \|T'(s)\|$ is called the curvature of γ at the point $\gamma(s)$. $\kappa(s) \geq 0$. If $\kappa(s) > 0$, $T'(s) / \kappa(s) = N = N(s)$ is called the **normal vector** or the **unit normal**.



Comment: Clockwise and counter-clockwise circles will share the same $\kappa(s)$ value, but different normals.

Definition 4.15

The scalar value $1/\kappa$ is called the radius of curvature.

Question: What is the meaning of the radius of curvature? Any intuitive guess?

Lemma 4.16

Let $\beta(t)$ and $\alpha(t)$ be regular parametric representations.

Then $\frac{d}{dt}(\alpha(t), \beta(t)) = (\alpha'(t), \beta(t)) + (\alpha(t), \beta'(t)).$

Proof

If $\alpha(t) = (x_\alpha, y_\alpha, z_\alpha)$ and $\beta(t) = (x_\beta, y_\beta, z_\beta)$, then

$(\alpha(t), \beta(t)) = (x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta)$. Differentiating,

$$\frac{d}{dt}(\alpha(t), \beta(t)) = x'_\alpha x_\beta + y'_\alpha y_\beta + z'_\alpha z_\beta + x_\alpha x'_\beta + y_\alpha y'_\beta + z_\alpha z'_\beta$$

$$= (\alpha'(t), \beta(t)) + (\alpha(t), \beta'(t)).$$

Lemma 4.17

If $\|\alpha(t)\| = 1$ for all values of t, then $(\alpha(t), \alpha'(t)) = 0$.

That is, $\alpha(t)$ is perpendicular to its derivative $\alpha'(t)$.

Proof

$$1 = \|\alpha(t)\|^2 = (\alpha(t), \alpha(t)),$$

so

$$\frac{d}{dt}(1) = 0 = \frac{d}{dt}(\alpha(t), \alpha(t))$$
$$= 2(\alpha(t), \alpha'(t)).$$

Theorem 4.18

$(T, N) = 0$. That is, the unit tangent and the unit normal are perpendicular to each other for all s.

Proof

Since, $\|T(s)\|^2 = 1 = (T(s), T(s))$,

we obtain from Lemma 4.17 above

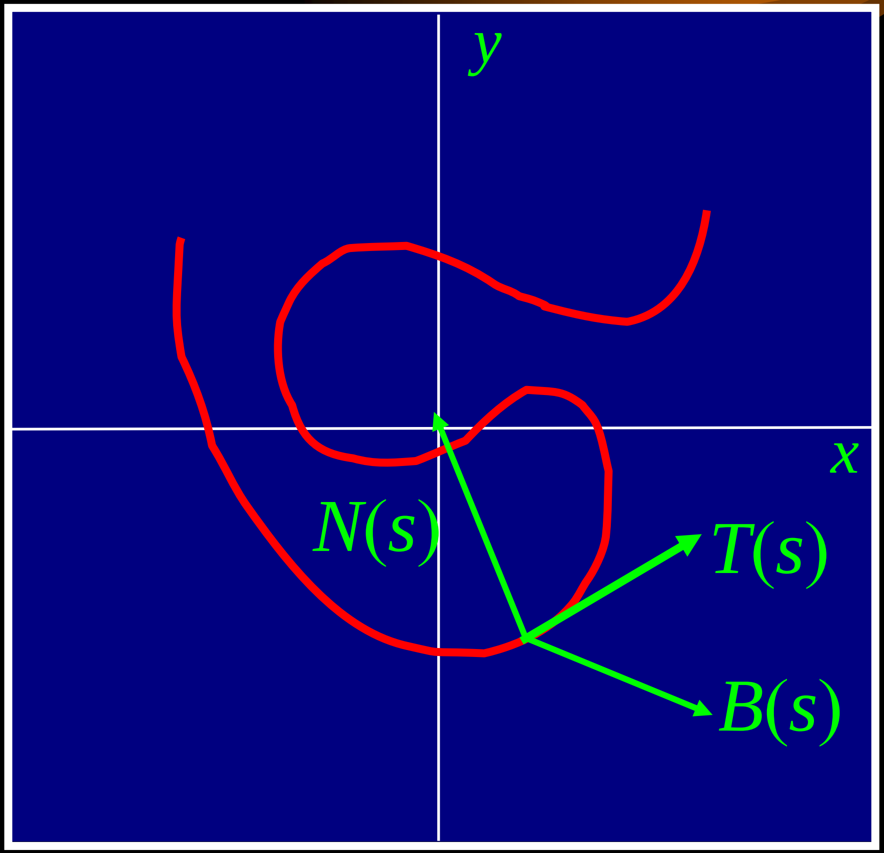
$$0 = 2(T(s), T'(s)) = 2k(T(s), N(s))$$

Definition 4.19

Finally we define the **binormal**,

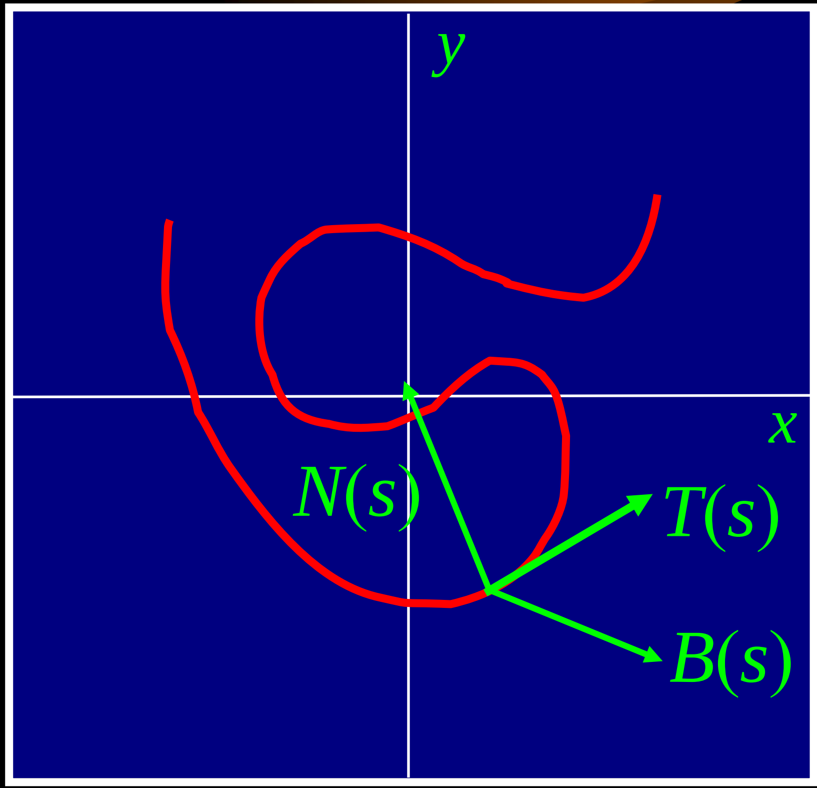
$$B(s) = T(s) \times N(s).$$

Because T and N are orthonormal, the triple $\{T, N, B\}$ forms a right-handed orthogonal set.



Definition 4.20

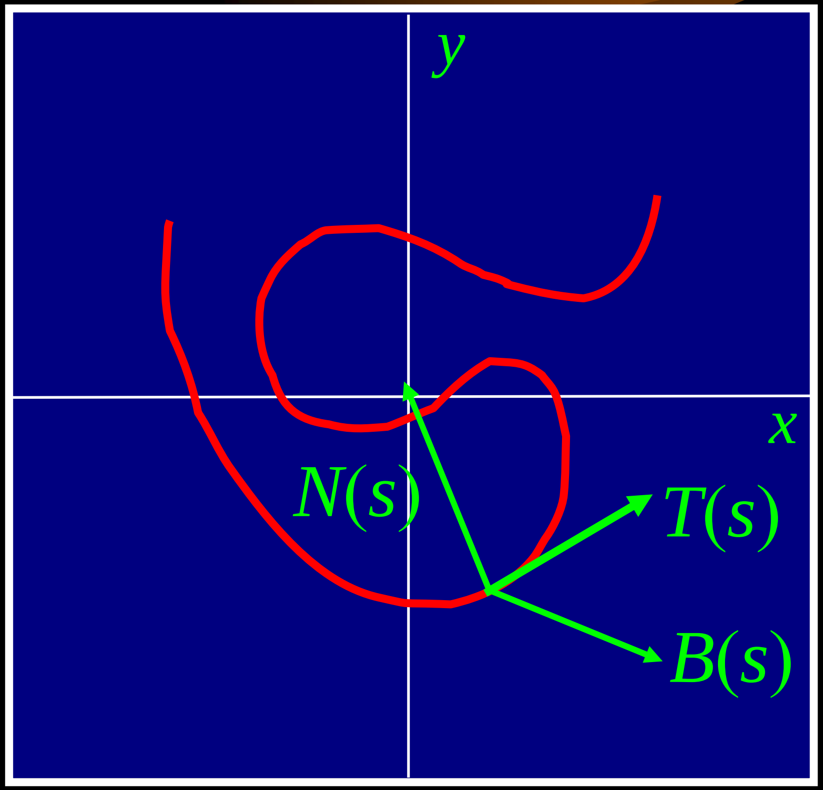
The triple $\{T, N, B\}$ forms an orthonormal basis for R^3 that changes for each point on the curve. It is called a **moving trihedron**.



Definition 4.21

The plane through the curve $\gamma(s)$ orthogonal to the curve normal N is called the

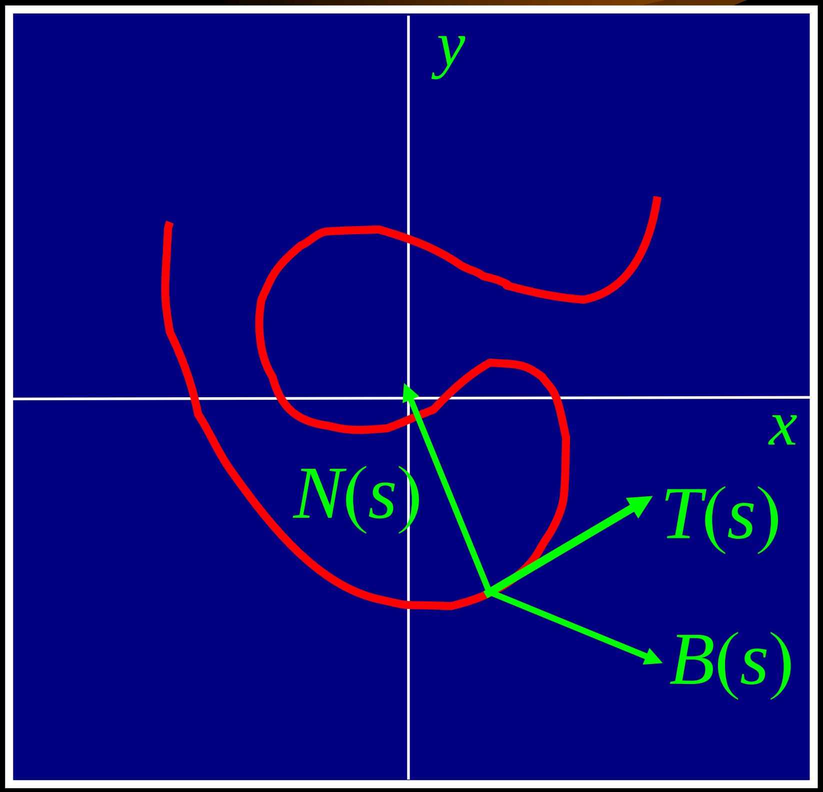
rectifying plane



Definition 4.22

The plane through the curve $\gamma(s)$ orthogonal to the curve normal B is called the

osculating plane

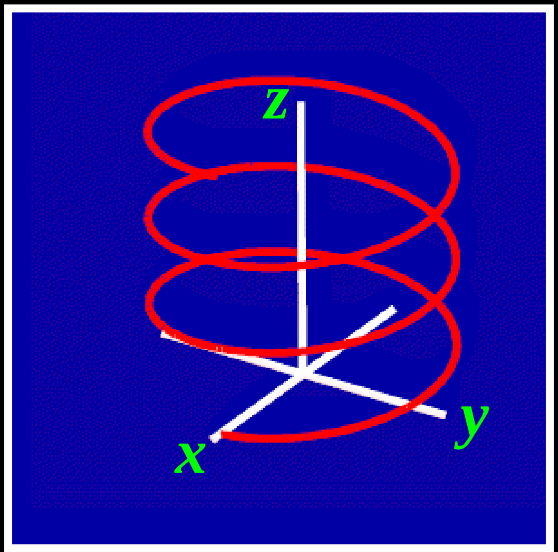


Example 4.23

For arbitrary a, b with $a > 0$, let $c = \sqrt{a^2 + b^2}$ and

define $\gamma(s) = \left(a\cos(s/c), a\sin(s/c), \dfrac{bs}{c} \right)$.

$(a\cos(s/c))^2 + (a\sin(s/c))^2 = a^2$ or the x and y coordinates lie on a circle of radius a. Hence $\gamma(s)$ lies on a cylinder. Because the z coordinate is linear in s, $\gamma(s)$ is a **helix**.



$$\gamma(s) = \left(a\cos(s/c), a\sin(s/c), \frac{bs}{c} \right).$$

Example 4.23 (Cont.)

Differentiating:

$$\gamma'(s) = \left(-\frac{a}{c}\sin(s/c), \frac{a}{c}\cos(s/c), \frac{b}{c} \right)$$

or

$$\begin{aligned}
\|\gamma'(s)\| &= \sqrt{(-a/c\sin(s/c))^2 + (a/c\cos(s/c))^2 + (b/c)^2} \\
&= \sqrt{(a/c)^2 + (b/c)^2} \\
&= \frac{\sqrt{a^2+b^2}}{c} \\
&= 1 \quad \text{and } \gamma(s) \text{ has arc length parameterization.}
\end{aligned}$$

$$\gamma'(s) = \left(-\frac{a}{c}\sin(s/c), \frac{a}{c}\cos(s/c), \frac{b}{c} \right)$$

Example 4.23 (Cont.)

The tangent vector equals:

$$T(s) = \left(-\frac{a}{c}\sin(s/c), \frac{a}{c}\cos(s/c), \frac{b}{c} \right).$$

Differentiating again,

$$\gamma''(s) = T'(s) = \left(-\frac{a}{c^2}\cos(s/c), -\frac{a}{c^2}\sin(s/c), 0 \right).$$

$$\gamma''(s) = T'(s) = \left(-\frac{a}{c^2}\cos(s/c), -\frac{a}{c^2}\sin(s/c), 0\right).$$

Example 4.23 (Cont.)

$$k(s) = \left\|T'(s)\right\| = \sqrt{\left(-\frac{a}{c^2}\cos(s/c)\right)^2 + \left(-\frac{a}{c^2}\sin(s/c)\right)^2} = \frac{\sqrt{a^2}}{c^2}$$

Or

$$k(s) = \frac{\sqrt{a^2}}{c^2} = \frac{a}{c^2} = \frac{a}{a^2+b^2} \text{ and if } b=0,\ k(s) = \frac{1}{a}.$$

Either way $\kappa(s)$ is constant.

$$\gamma''(s) = T'(s) = \left(-\frac{a}{c^2}\cos(s/c), -\frac{a}{c^2}\sin(s/c), 0 \right).$$

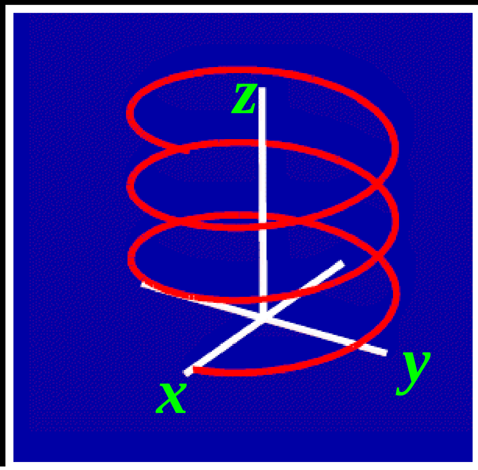
Example 4.23 (Cont.)

Because $T'(s) = k(s)\, N(s)$, $N(s) = (-\cos(s/c), -\sin(s/c), 0)$, or the normal always points towards the cylinder's axis.

Finally the binormal,

$$B = T \times N = \left(-\frac{a}{c}\sin(s/c), \frac{a}{c}\cos(s/c), \frac{b}{c} \right) \times \left(-\cos(s/c), -\sin(s/c), 0 \right)$$

$$= \left(\frac{b}{c}\sin(s/c), -\frac{b}{c}\cos(s/c), \frac{a}{c} \right).$$



The Frenet Equations

Since $\{ T(s), N(s), B(s) \}$ form a right-handed orthonormal triple, they also form a basis for R^3. Specifically, we should be able to express $T'(s)$, $N'(s)$, and $B'(s)$ as a linear combination of $\{ T(s), N(s), B(s) \}$:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

The Frenet Equations

We know, by now, the following:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

because $(V(s), V'(s)) = 0$ for any unit size vector field.

The Frenet Equations (Cont.)

Because $B = T \times N$, $(B, T) = 0$, or
$$0 = \frac{d}{ds}\left(B(s), T(s)\right)$$
$$= \left(B', T\right) + \left(B, T'\right)$$
$$= \left(B', T\right) + \kappa\left(B, N\right)$$
$$= \left(B', T\right),$$

and then,
$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ & 0 & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

The Frenet Equations (Cont.)

So B' is in the direction of N. Let $B' = -\tau(s)N(s)$. τ is called the torsion of the curve.

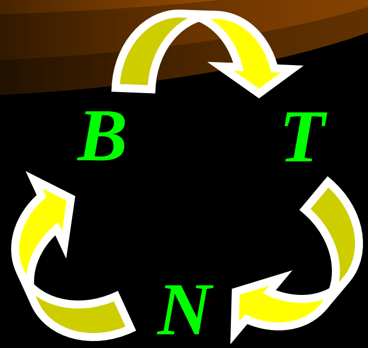
Question: What is the meaning of the torsion?

Question: What is the torsion of a planar curve?

Now, we have, $$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ & 0 & \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

The Frenet Equations (Cont.)

$B = T \times N$ and so $N = B \times T$. One can show that $N' = B' \times T + B \times T'$. Then,



$$
\begin{aligned}
N' &= B' \times T + B \times T' \\
&= (-\tau N) \times T + B \times (\kappa N) \\
&= -\tau (N \times T) + \kappa (B \times N) \\
&= \tau B - \kappa T \\
&= -\kappa T + \tau B.
\end{aligned}
$$

Theorem 4.25 (Frenet Equations)

For an arc length parameterized curve $\gamma(s)$ with unit tangent $T(s)$, normal $N(s)$, binormal $B(s)$, curvature $\kappa(s)$, and torsion $\tau(s)$:

$$
\begin{aligned}
T'(s) &= \kappa(s)N(s) \\
N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\
B'(s) &= -\tau(s)N(s)
\end{aligned}
$$

This can also be written in matrix form as,

$$
\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.
$$

$$B = \left(\frac{b}{c}\sin(s/c), -\frac{b}{c}\cos(s/c), \frac{a}{c} \right)$$

Example 4.26

For our helix,

$$B^{'}(s) = \left(\frac{b}{c^2}\cos(s/c), \frac{b}{c^2}\sin(s/c), 0 \right)$$
$$= -\tau(s)(-\cos(s/c), -\sin(s/c), 0),$$

and $\tau(s) = b/c^2$.

Lemma 4.27

If $\gamma(s)$ is an arc length parameterized curve such that τ is identically 0 and such that κ is a constant, then $\gamma(s)$ is a circle.

Proof

Define $\alpha = \gamma + (1/\kappa)N$. Then since, $\tau = 0$,

$$\alpha' = \gamma' + \left(\frac{1}{\kappa}\right)' N + \left(\frac{1}{\kappa}\right) N'$$

$$= T + 0N + \left(\frac{1}{\kappa}\right)\left[-\kappa T + \tau B\right]$$

$$= T - T = 0.$$

Lemma 4.27 (Cont.)

If $\alpha' = 0$, $\alpha = c$, a constant. Then,

$$\|\alpha - \gamma\| = \| c - \gamma \| = \|(1/\kappa)N\| = |1/\kappa|.$$

Thus, all points on γ are at distance $1/\kappa$ from a specified point, c. In other words, γ is a circle by definition.

Question: What is γ, if γ is an arc length parameterized curve such that τ and κ are both constant?

Question: What if $\kappa = 0$?

Frenet Frame Field

The triple $\{\ T(s), N(s), B(s)\ \}$ is known as the

Frenet Frame Field.

The osculating plane (spanned by $\{\ T(s), N(s)\ \}$) contains "more" curve than any other plane.

Taylor Expansion

A second order Taylor expansion of $\gamma(s)$ yields:

$$\begin{aligned}\gamma_T(s) &= \gamma(s_0) + \gamma'(s_0)(s - s_0) + \gamma''(s_0)(s - s_0)^2 / 2 \\ &= \gamma(s_0) + T(s_0)(s - s_0) + \kappa(s_0)N(s_0)(s - s_0)^2 / 2.\end{aligned}$$

Now consider the third derivative of $\gamma(s)$,

$$\gamma'''(s_0) = (\kappa N)' = \frac{d\kappa}{ds}(s_0)N + \kappa(-\kappa T + \tau B) = \frac{d\kappa}{ds}(s_0)N - \kappa^2 T + \kappa\tau B\,.$$

$$\gamma_T(s) = \gamma(s_0) + T(s_0)(s - s_0) + \kappa(s_0)N(s_0)(s - s_0)^2 / 2$$

$$\gamma'''(s_0) = \frac{d\kappa}{ds}(s_0)N - \kappa^2 T + \kappa\tau B.$$

Taylor Expansion (Cont.)

A third order Taylor expansion of $\gamma(s)$ yields,

$$\gamma_T(s) = \gamma(s_0) + T(s_0)\left[(s - s_0) - \frac{\kappa^2 (s - s_0)^3}{6}\right] + N(s_0)\left[\frac{\kappa(s - s_0)^2}{2} + \frac{d\kappa}{ds}(s_0)\frac{(s - s_0)^3}{6}\right] + B(s_0)\left[\frac{\kappa\tau(s - s_0)^3}{6}\right].$$

Osculating Circle

Consider the parametric circular form of,

$$c(s) = \boldsymbol{k} + r\cos(s/r)\,\boldsymbol{k}_1 + r\sin(s/r)\,\boldsymbol{k}_2,$$

centered at $\boldsymbol{k}$ and with $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ two orthogonal unit vectors.

Question: What is the best fitting circle to $\gamma(s)$ at $s = s_0$?

Second order fitting entails the following three constraints:

$$c(s - s_0) = \gamma(s_0)$$
$$c'(s - s_0) = \gamma'(s_0)$$
$$c''(s - s_0) = \gamma''(s_0)$$

$$c(s) = \boldsymbol{k} + r\cos(s/r)\,\boldsymbol{k}_1 + r\sin(s/r)\,\boldsymbol{k}_2$$

Osculating Circle (Cont.)

Differentiating $c(s)$,

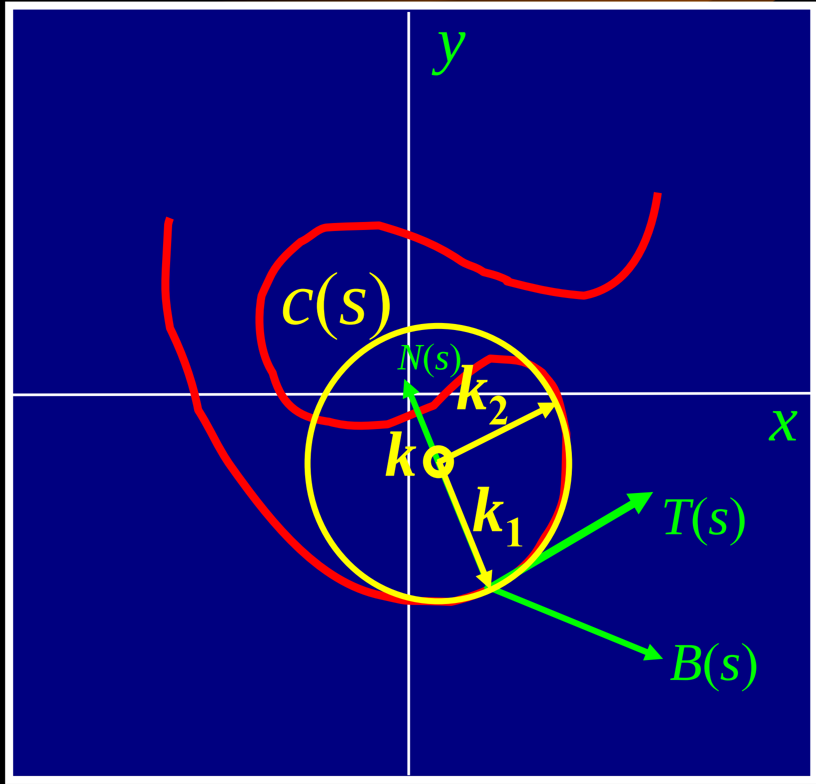
$$c(0) = \gamma(s_0) = \boldsymbol{k} + r\boldsymbol{k}_1$$

$$c'(0) = \gamma'(s_0) = T = \boldsymbol{k}_2$$

$$c''(0) = \gamma''(s_0) = \kappa N = -\frac{1}{r}\boldsymbol{k}_1$$

or $\boldsymbol{k}_1 = -N$, $\boldsymbol{k}_2 = T$, $r = 1/\kappa$,

$$\boldsymbol{k} = \gamma(s_0) - r\boldsymbol{k}_1.$$



Theorem 4.31 (The Fundamental Theorem of the Differential Geometry of Curves)

Let $k(s)$ and $f(s)$ be continuous scalar valued functions on $a \leq s \leq b$, and $0 < k(s)$.

There exists exactly **one regular curve** $\gamma(s)$ for which $k(s)$ is the curvature and $f(s)$ is the torsion.

The position of the curve in space is, however, free.

These functions are called the **intrinsic equations** of the curve.

Proof

Existence of a solution curve, given $k(s)$ and $f(s)$, is a direct result of the existence of a solution to the given set of differential equations and is out of our scope.

Uniqueness can be proven by contradiction. Assume $\alpha(s)$ and $\beta(s)$ both satisfy $\kappa_\alpha(s) = \kappa_\beta(s)$ and $\tau_\alpha(s) = \tau_\beta(s)$, $s \in \mathbf{I}$, and let $T_\alpha{}^0, N_\alpha{}^0, B_\alpha{}^0$ and $T_\beta{}^0, N_\beta{}^0, B_\beta{}^0$ be the Frenet trihedrons at $s = s_0$. Further, using rigid motion, one can coerce $\alpha(s_0) = \beta(s_0)$, and $T_\alpha{}^0 = T_\beta{}^0$, $N_\alpha{}^0 = N_\beta{}^0$, $B_\alpha{}^0 = B_\beta{}^0$.

Proof (Cont.)

We now have

$$\frac{dT_\alpha}{ds} = \kappa N_\alpha \qquad \frac{dT_\beta}{ds} = \kappa N_\beta$$

$$\frac{dN_\alpha}{ds} = -\kappa T_\alpha + \tau B_\alpha \qquad \frac{dN_\beta}{ds} = -\kappa T_\beta + \tau B_\beta$$

$$\frac{dB_\alpha}{ds} = -\tau N_\alpha \qquad \frac{dB_\beta}{ds} = -\tau N_\beta$$

as the Frenet equations of the two curves.

We also have the initial conditions of

$\alpha(s_0)=\beta(s_0)$, and $T_\alpha{}^0 = T_\beta{}^0, N_\alpha{}^0 = N_\beta{}^0, B_\alpha{}^0 = B_\beta{}^0.$

Proof (Cont.)

Consider

$$\frac{1}{2}\frac{d}{ds}\left\{\left\|T_\alpha - T_\beta\right\|^2 + \left\|N_\alpha - N_\beta\right\|^2 + \left\|B_\alpha - B_\beta\right\|^2\right\}$$

$$= \left\langle T_\alpha - T_\beta, T_\alpha' - T_\beta'\right\rangle + \left\langle B_\alpha - B_\beta, B_\alpha' - B_\beta'\right\rangle + \left\langle N_\alpha - N_\beta, N_\alpha' - N_\beta'\right\rangle$$

$$= \kappa\left\langle T_\alpha - T_\beta, N_\alpha - N_\beta\right\rangle - \tau\left\langle B_\alpha - B_\beta, N_\alpha - N_\beta\right\rangle$$

$$-\kappa\left\langle N_\alpha - N_\beta, T_\alpha - T_\beta\right\rangle + \tau\left\langle N_\alpha - N_\beta, B_\alpha - B_\beta\right\rangle$$

$$= 0.$$

Proof (Cont.)

Hence $\Delta(s) = \left\| T_\alpha - T_\beta \right\|^2 + \left\| N_\alpha - N_\beta \right\|^2 + \left\| B_\alpha - B_\beta \right\|^2$ is constant!

But $\Delta(s)$ is known to be zero for $s = s_0$ and, therefore, is zero throughout. It follows that $T_\alpha = T_\beta$, $N_\alpha = N_\beta$, $B_\alpha = B_\beta$ for all $s \in \mathbf{I}$.

Proof (Cont.)

Because $T_\alpha = T_\beta$, $\alpha'(s) = \beta'(s)$ or $\alpha(s) - \beta(s) = c$.

Because $\alpha(s_0) = \beta(s_0)$, we conclude that $c = 0$.

Therefore, $\alpha(s) \equiv \beta(s)$, for all $s \in \mathbf{I}$.

Question: if $T_\alpha^0 = T_\beta^0$ and $N_\alpha^0 = N_\beta^0$, what about B_α^0? B_β^0?

Frenet Equations for Non Arc Length Parameterizations

Let $\beta(t)$ be a regular curve. Then, $s = \int_{t_0}^{t} \| \beta'(w) \| \, dw$ and $\frac{ds}{dt} = \| \beta'(t) \|$, yielding:

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \kappa N \left\| \beta'(t) \right\|,$$

$$\frac{dN}{dt} = \frac{dN}{ds}\frac{ds}{dt} = \left(-\kappa T + \tau B\right) \left\| \beta'(t) \right\|,$$

$$\frac{dB}{dt} = \frac{dB}{ds}\frac{ds}{dt} = -\tau N \left\| \beta'(t) \right\|.$$

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Consider the unit tangent vector, T, of $\beta(t)$,

$$T = \frac{d\beta(t)}{ds} = \frac{d\beta(t)}{dt}\frac{dt}{ds} = \frac{\beta'(t)}{\left(\frac{ds}{dt}\right)} = \frac{\beta'(t)}{\|\beta'(t)\|},$$

so, $T_\beta(t) = \dfrac{\beta'(t)}{\|\beta'(t)\|}$ and $\beta'(t) = \|\beta'(t)\| T_\beta.$

Question: β' is in the direction of T. What is the direction of β''?

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Consider the second derivative of $\beta(t)$,

$$\beta''(t) = \frac{d\beta'(t)}{dt}$$

$$= \left(\frac{d\|\beta'(t)\|}{dt}\right)T(t) + \left(\frac{dT(t)}{dt}\right)\|\beta'(t)\| = \left(\frac{d\|\beta'(t)\|}{dt}\right)T(t) + \left(\frac{dT(t)}{ds}\frac{ds}{dt}\right)\|\beta'(t)\|$$

$$= \left(\frac{d\|\beta'(t)\|}{dt}\right)T(t) + \left(\frac{dT(t)}{ds}\right)\|\beta'(t)\|^2 = \left(\frac{d\|\beta'(t)\|}{dt}\right)T(t) + (\kappa(t)N(t))\|\beta'(t)\|^2.$$

Hence β'' is no longer in the direction of N.

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Question: What is the direction of $\beta' \times \beta''$?

$$\beta' \times \beta'' = \left(\left\|\beta'\right\| T(t)\right) \times \left[\left(\frac{d\left\|\beta'(t)\right\|}{dt}\right) T(t) + \kappa(t) N(t) \left\|\beta'(t)\right\|^2\right]$$

$$= \left\|\beta'\right\| \left(\frac{d\left\|\beta'(t)\right\|}{dt}\right) \left(T(t) \times T(t)\right) + \left\|\beta'\right\|^3 \kappa(t) \left(T(t) \times N(t)\right)$$

$$= \left\|\beta'\right\|^3 \kappa(t) B(t).$$

Frenet Equations for Non Arc Length Parameterizations (Cont.)

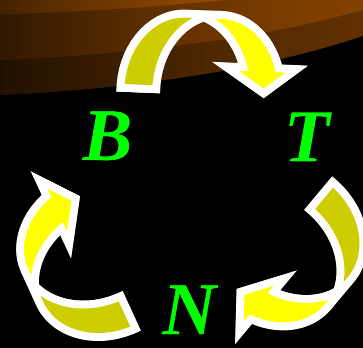
Because $\beta' \times \beta'' = \|\beta'\|^3 \kappa(t) B$, we have $B = \dfrac{\beta' \times \beta''}{\|\beta' \times \beta''\|}$,

and, $\|\beta' \times \beta''\| = \|\beta'\|^3 \kappa(t) \|B\| = \|\beta'\|^3 \kappa(t)$,

or $\kappa(t) = \dfrac{\|\beta' \times \beta''\|}{\|\beta'\|^3}$.

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Now $N = B \times T$ and hence,



$$N = \frac{\beta'(t) \times \beta''(t)}{\left\| \beta'(t) \times \beta''(t) \right\|} \times \frac{\beta'}{\left\| \beta' \right\|}$$

$$= \frac{\left(\beta'(t) \times \beta''(t) \right) \times \beta'}{\left\| \beta' \right\| \left\| \beta' \times \beta'' \right\|}$$

$$= \frac{\left(\beta'(t), \beta'(t) \right) \beta''(t)}{\left\| \beta' \right\| \left\| \beta' \times \beta'' \right\|} - \frac{\left(\beta'(t), \beta''(t) \right) \beta'(t)}{\left\| \beta' \right\| \left\| \beta' \times \beta'' \right\|}.$$

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Finally, we need to express τ in the given parameterization.

In order to compute τ, we seek the third derivative of $\beta(t)$.

Yet, because $\{ T, N, B \}$ is an orthonormal system,

$$\beta''' = (\beta''', T)T + (\beta''', N)N + (\beta''', B)B.$$

Frenet Equations for Non Arc Length Parameterizations (Cont.)

$$\beta''(t) = \left(\frac{d\left\|\beta'(t)\right\|}{dt}\right)T(t) + \left(\kappa(t)N(t)\right)\left\|\beta'(t)\right\|^2$$

Consider the third derivative of $\beta(t)$,

$$\beta''' = \left(\frac{d\left\|\beta'\right\|}{dt}T + \kappa\left\|\beta'\right\|^2 N\right)' = \left(\frac{d\left\|\beta'\right\|}{dt}T\right)' + \left(\kappa\left\|\beta'\right\|^2 N\right)'$$

$$= \frac{d^2\left\|\beta'\right\|}{dt^2}T + \frac{d\left\|\beta'\right\|}{dt}\frac{dT}{dt} + \left(\kappa\left\|\beta'\right\|^2\right)' N + \left(\kappa\left\|\beta'\right\|^2\right)\frac{dN}{dt}$$

$$= \frac{d^2\left\|\beta'\right\|}{dt^2}T + \frac{d\left\|\beta'\right\|}{dt}\left\|\beta'\right\|\kappa N + \left(\kappa\left\|\beta'\right\|^2\right)' N + \left(\kappa\left\|\beta'\right\|^2\right)\left\|\beta'\right\|\left(-\kappa T + \tau B\right)$$

$$= \left(\frac{d^2\left\|\beta'\right\|}{dt^2} - \kappa^2\left\|\beta'\right\|^3\right)T + \left(\kappa\frac{d\left\|\beta'\right\|}{dt}\left\|\beta'\right\| + \left(\kappa\left\|\beta'\right\|^2\right)'\right)N + \kappa\tau\left\|\beta'\right\|^3 B.$$

Frenet Equations for Non Arc Length Parameterizations (Cont.)

Then, $\left(\beta''', B\right) = \kappa\tau\left\|\beta'\right\|^3$ and we have,

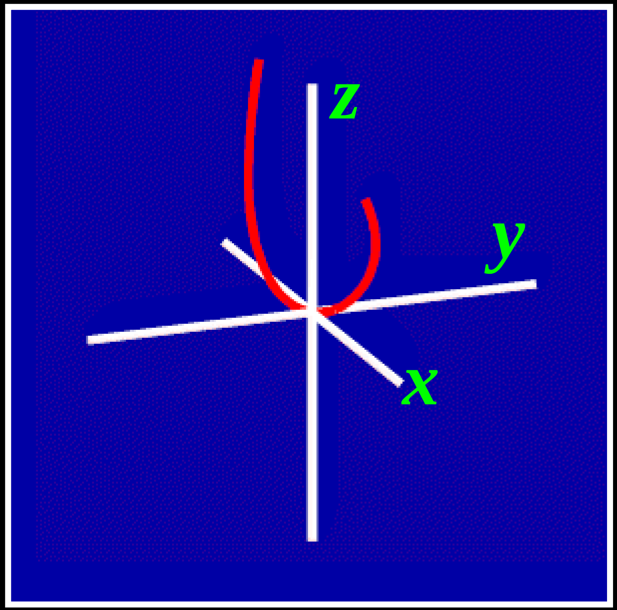
$$\tau = \frac{\left(\beta''', B\right)}{\kappa\left\|\beta'\right\|^3} = \frac{\left(\beta''', \dfrac{\beta' \times \beta''}{\left\|\beta' \times \beta''\right\|}\right)}{\left\|\beta' \times \beta''\right\|} = \frac{\left(\beta''', \beta' \times \beta''\right)}{\left\|\beta' \times \beta''\right\|^2}$$

recalling that $\kappa = \dfrac{\left\|\beta' \times \beta''\right\|}{\left\|\beta'\right\|^3}$ and $B = \dfrac{\beta' \times \beta''}{\left\|\beta' \times \beta''\right\|}$.

Example 4.32

Consider the curve $\beta(t) = (3t - t^3, 3t^2, 3t+t^3)$. We seek the moving trihedron as well as the curvature and torsion of β.

$\beta'(t) = (3 - 3t^2, 6t, 3+3t^2)$ and clearly $||\beta'(t)|| \neq 1$, or not an arc length parameterized curve.



Example 4.32 (Cont.)

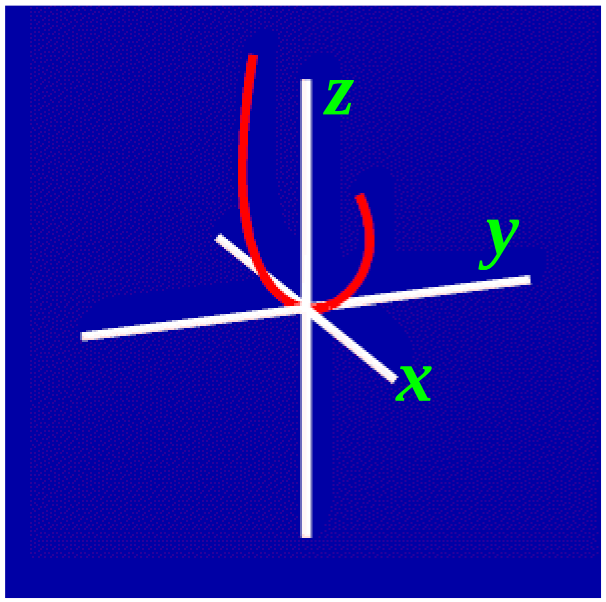
The derivatives of

$\beta(t) = (3t - t^3, 3t^2, 3t+t^3)$ yields,

$$\beta'(t) = 3(1 - t^2, 2t, 1 + t^2),$$

$$\beta''(t) = 6(-t, 1, t),$$

$$\beta'''(t) = 6(-1, 0, 1),$$



and $(\beta', \beta') = 9\left((1 - t^2)^2 + 4t^2 + (1 + t^2)^2\right) = 18(1 + t^2)^2$

or $\|\beta'\| = (1 + t^2)\sqrt{18}.$

Example 4.32 (Cont.)

$$\beta'(t) = 3(1-t^2, 2t, 1+t^2),$$
$$\beta''(t) = 6(-t, 1, t),$$
$$\beta'''(t) = 6(-1, 0, 1),$$
$$\|\beta'\| = (1+t^2)\sqrt{18}.$$

The tangent field equals $T = \dfrac{(1-t^2, 2t, 1+t^2)}{\sqrt{2}(1+t^2)}.$

$$\beta' \times \beta'' = 18\begin{vmatrix} e_1 & e_2 & e_3 \\ (1-t^2) & 2t & 1+t^2 \\ -t & 1 & t \end{vmatrix} = 18(-1+t^2, -2t, 1+t^2),$$

and

$$\|\beta' \times \beta''\|^2 = 18^2\left[(-1+t^2)^2 + 4t^2 + (1+t^2)^2\right] = 2(18)^2(1+t^2)^2.$$

Example 4.32 (Cont.)

$$\beta'(t) = 3(1-t^2, 2t, 1+t^2),$$
$$\beta''(t) = 6(-t, 1, t),$$
$$\beta'''(t) = 6(-1, 0, 1),$$
$$\|\beta'\| = (1+t^2)\sqrt{18}.$$
$$\beta' \times \beta'' = 18(-1+t^2, -2t, 1+t^2),$$

Then, $$B = \frac{\beta' \times \beta''}{\|\beta' \times \beta''\|} = \frac{(-1+t^2, -2t, 1+t^2)}{\sqrt{2}(1+t^2)}.$$

Now $$(\beta' \times \beta'', \beta''') = 6 \cdot 18((1-t^2) + 0 + (1+t^2)) = 6 \cdot 18(2)$$

and $$\tau = \frac{(\beta' \times \beta'', \beta''')}{\|\beta' \times \beta''\|^2} = \frac{12 \cdot 18}{2 \cdot 18^2 (1+t^2)^2} = \frac{1}{3(1+t^2)^2}.$$

Example 4.32 (Cont.)

$$\beta'(t) = 3(1-t^2, 2t, 1+t^2),$$
$$\beta''(t) = 6(-t, 1, t),$$
$$\beta'''(t) = 6(-1, 0, 1),$$
$$\|\beta'\| = (1+t^2)\sqrt{18}.$$
$$\beta' \times \beta'' = 18(-1+t^2, -2t, 1+t^2),$$

Finally,

$$\kappa = \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3} = \frac{18\sqrt{2}(1+t^2)}{(1+t^2)^3 18^{3/2}} = \frac{1}{3(1+t^2)^2}.$$

Example 4.33

Consider $\gamma(t) = (t, f(t), 0)$, a $C^{(2)}$ continuous function.

Compute the curvature of $\gamma(t)$:

$$\gamma'(t) = \left(1, f'(t), 0\right)$$
$$\gamma''(t) = \left(0, f''(t), 0\right)$$

or

$$\kappa(t) = \frac{\left\|\gamma'(t) \times \gamma''(t)\right\|}{\left\|\gamma'(t)\right\|^3} = \frac{\left\| f''(t)\right\|}{\left(\sqrt{1 + (f'(t))^2}\right)^3}$$

Piecing Together Parametric Curves

Consider two regular curves $\gamma_1(t)$ and $\gamma_2(t)$, and let

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0,1) \\ \gamma_2(t-1) & t \in [1,2]. \end{cases}$$

We investigate when $\gamma(t)$ is $C^{(-1)}$, $C^{(0)}$, $C^{(1)}$, $C^{(2)}$, or curvature continuous at $t=1$.

Definition 4.34 (Geometric Continuity)

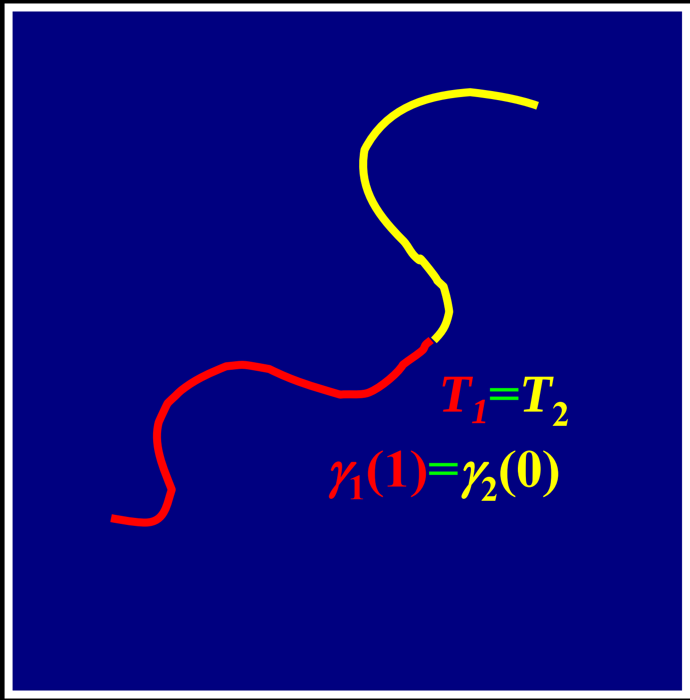
The term geometrically $C^{(n)}$ will mean that a curve is $C^{(n)}$ when parameterized in the arc length parameterization. Geometrically $C^{(n)}$ is also denoted $G^{(n)}$.

Geometrically $C^{(1)}$ means that the unit tangents are continuous.

Geometric Continuity (Cont.)

$G^{(0)}$ is simple to check, requiring $\gamma_1(1)=\gamma_2(0)$. For $G^{(1)}$

$$T_1 = \frac{\gamma_1'(1)}{\left\|\gamma_1'(1)\right\|} = \frac{\gamma_2'(0)}{\left\|\gamma_2'(0)\right\|} = T_2.$$



Let T denote the tangent vector at $\gamma(1)$. Denote by

$c_1 = \|\gamma_1'(1)\|$,

$c_2 = \|\gamma_2'(0)\|$, and

$k_1 = c_1 / c_2$.

Geometric Continuity (Cont.)

Then $\gamma_1'(1) = k_1 \gamma_2'(0)$.

Question: What is changing in $\gamma(t)$, from $t = 1^-$ to $t = 1^+$?

Question: What is necessary for $\gamma(t)$ to be curvature continuous?

Question: if the unit tangent, T, and the unit normal, N, are continuous at $\gamma(1)$, what about B?

Geometric Continuity (Cont.)

Recall that $\kappa(t)B(t) = \dfrac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t)\|^3}$.

A necessary and sufficient condition for curvature continuity at $t = 1$, is that $\kappa_1(1)B_1(1) = \kappa_2(0)B_2(0)$, assuming $G^{(1)}$. Now,

$$T = \frac{\gamma'(1)}{\|\gamma'(1)\|} = \frac{\gamma_1'(1)}{\|\gamma_1'(1)\|} = \frac{\gamma_2'(0)}{\|\gamma_2'(0)\|}.$$

Geometric Continuity (Cont.)

We have, $\dfrac{\gamma_1'(1)\times\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^3} = \dfrac{\gamma_1'(1)}{\left\|\gamma_1'(1)\right\|} \times \dfrac{\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^2} = T \times \dfrac{\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^2},$

and also, $\dfrac{\gamma_2'(0)\times\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^3} = \dfrac{\gamma_2'(0)}{\left\|\gamma_2'(0)\right\|} \times \dfrac{\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^2} = T \times \dfrac{\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^2},$

yielding, $T \times \dfrac{\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^2} = T \times \dfrac{\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^2}.$

Geometric Continuity (Cont.)

From $T \times \frac{\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^2} = T \times \frac{\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^2}$ we have,

$$0 = T \times \left(\frac{\gamma_1''(1)}{\left\|\gamma_1'(1)\right\|^2} - \frac{\gamma_2''(0)}{\left\|\gamma_2'(0)\right\|^2} \right) = T \times \left(\gamma_1''(1) - \frac{\left\|\gamma_1'(1)\right\|^2}{\left\|\gamma_2'(0)\right\|^2} \gamma_2''(0) \right)$$

$$= T \times \left(\gamma_1''(1) - (k_1)^2 \gamma_2''(0) \right).$$

$$0 = T \times \left(\gamma_1''(1) - (k_1)^2 \gamma_2''(0)\right).$$

Geometric Continuity (Cont.)

A zero cross product suggests the two vectors point in the same direction. Hence,

$$c_3 \gamma_2'(0) = \gamma_1''(1) - (k_1)^2 \gamma_2''(0)$$

or,

$$\gamma_1''(1) = c_3 \gamma_2'(0) + (k_1)^2 \gamma_2''(0).$$

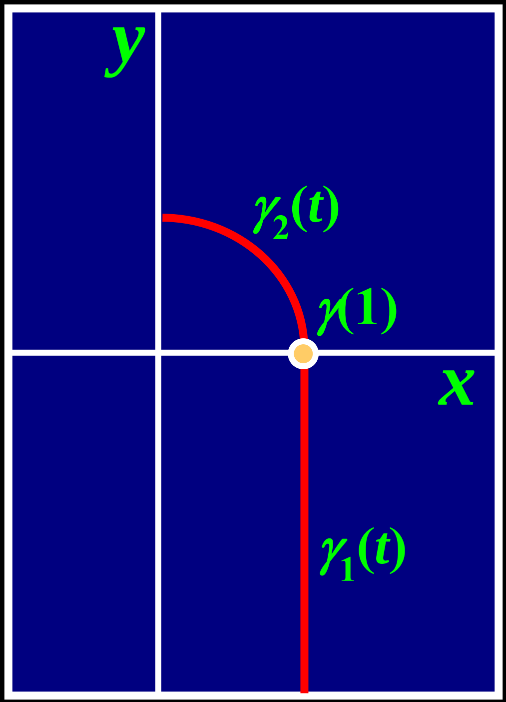
$\gamma_1''(1) = c_3\gamma_2'(0) + (k_1)^2\gamma_2''(0).$

Example 4.35

Let $\gamma_1(t) = (1, -5)(1-t) + (1, 0)t$, and $\gamma_2(t) = (\cos\pi t, \sin\pi t)$,

and let

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0,1) \\ \gamma_2(t-1) & t \in [1,2]. \end{cases}$$



Question: What is the continuity of $\gamma(t)$?

$\gamma_1'(1) = (0, 5)$, $\gamma_2'(0) = (0, \pi)$,

hence $G^{(1)}$ and $k_1 = 5 / \pi$.

$\gamma_1''(1) = (0, 0)$, $\gamma_2''(0) = (-\pi^2, 0)$,

hence not $C^{(2)}$.

$$\gamma_1''(1) = c_3\gamma_2'(0) + (k_1)^2\gamma_2''(0).$$

Example 4.35 (Cont.)

Further

$$c_3\gamma_2'(0) + (k_1)^2\gamma_2''(0) =$$
$$c_3(0, \pi) + \frac{25}{\pi^2}(-\pi^2, 0) =$$
$$(-25, \pi c_3).$$

that can never equal

$\gamma_1''(1) = (0, 0).$