Computer Aided Geometric Design

Shape Approximation

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based on a book by Cohen, Riesenfeld, & Elber

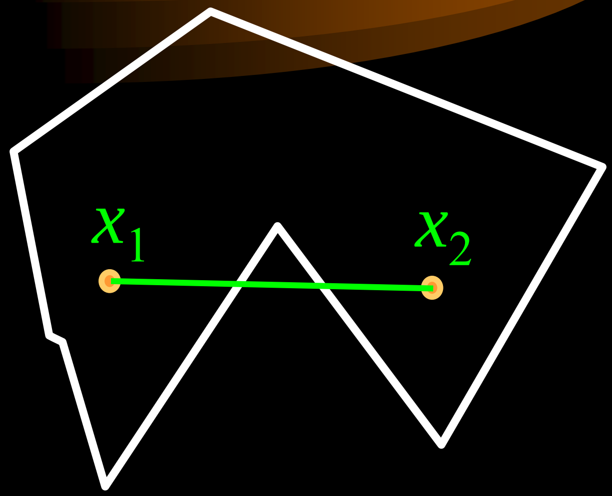
Shape Approximation

- Point Precision - Should we require that certain points be interpolated?
- Continuity - What level of continuity should we desire (smoothness?)
- Closeness - How could we measure the fidelity of a curve “close” to points?
- Good Fit - How does one decide what is a good fit?

Definition 1.35

A subset of $\boldsymbol{R}^3$, $\boldsymbol{X}$ is called convex if for all $x_1, x_2 \in \boldsymbol{X}$,

$$(1-t)\, x_1 + t\, x_2 \in \boldsymbol{X}, \quad t \in [0, 1].$$



That is, the line segment connecting x_1 and x_2 lies entirely within the set $\boldsymbol{X}$.

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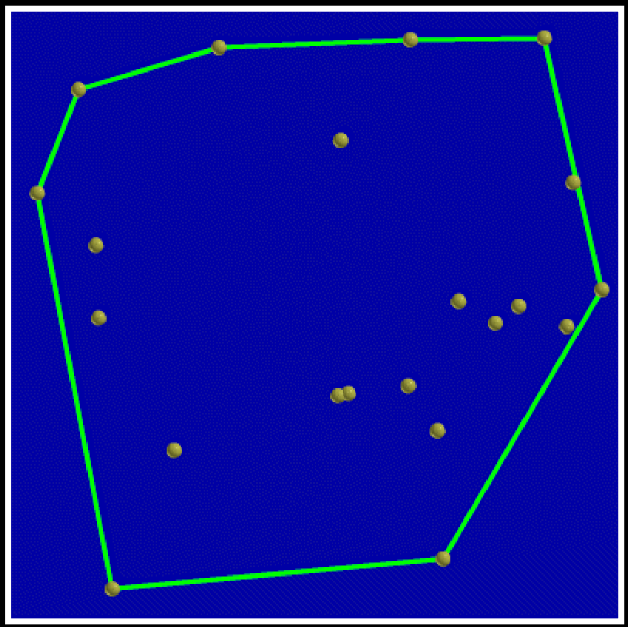
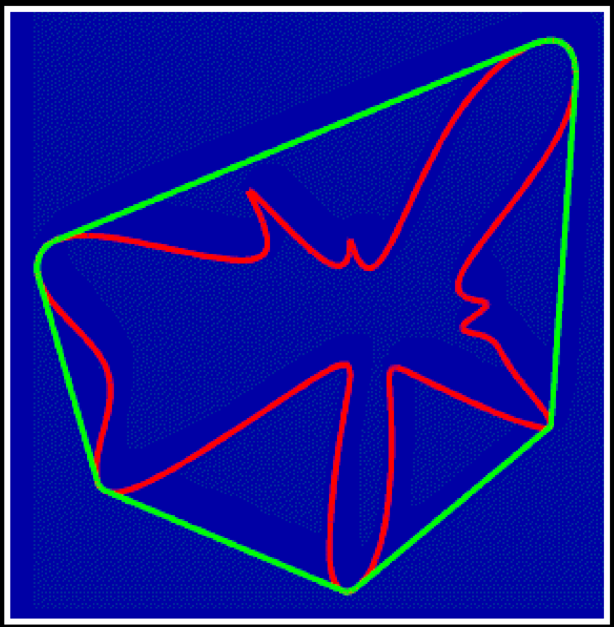
Definition 1.36

The convex hull of a set *X* is the smallest convex set containing *X*.



Convex Hull of **Points** (left)

Closed Curve (right)



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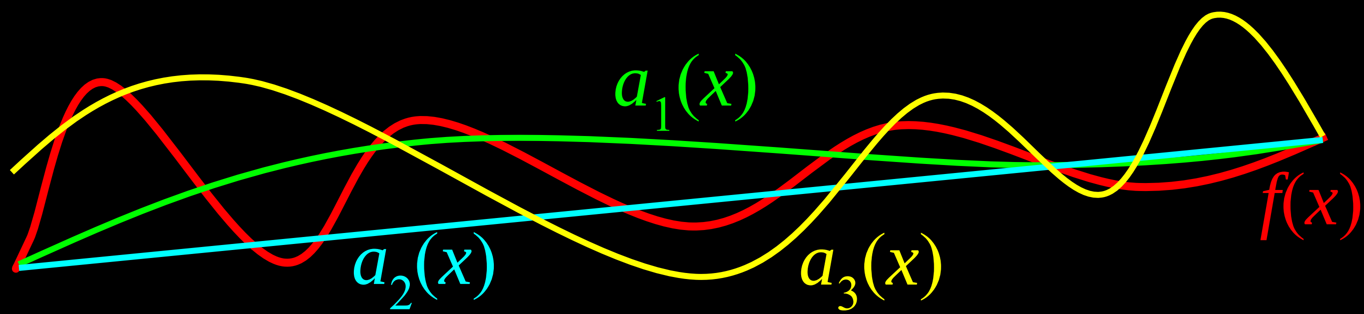
Definition 2.19

We introduce the notion of variation diminishing.

If $a(x)$ is an approximation to a function $f(x)$, a is called variation diminishing if a straight line can intersect $a(x)$ no more often than it intersects $f(x)$.



$\pi = 2$?

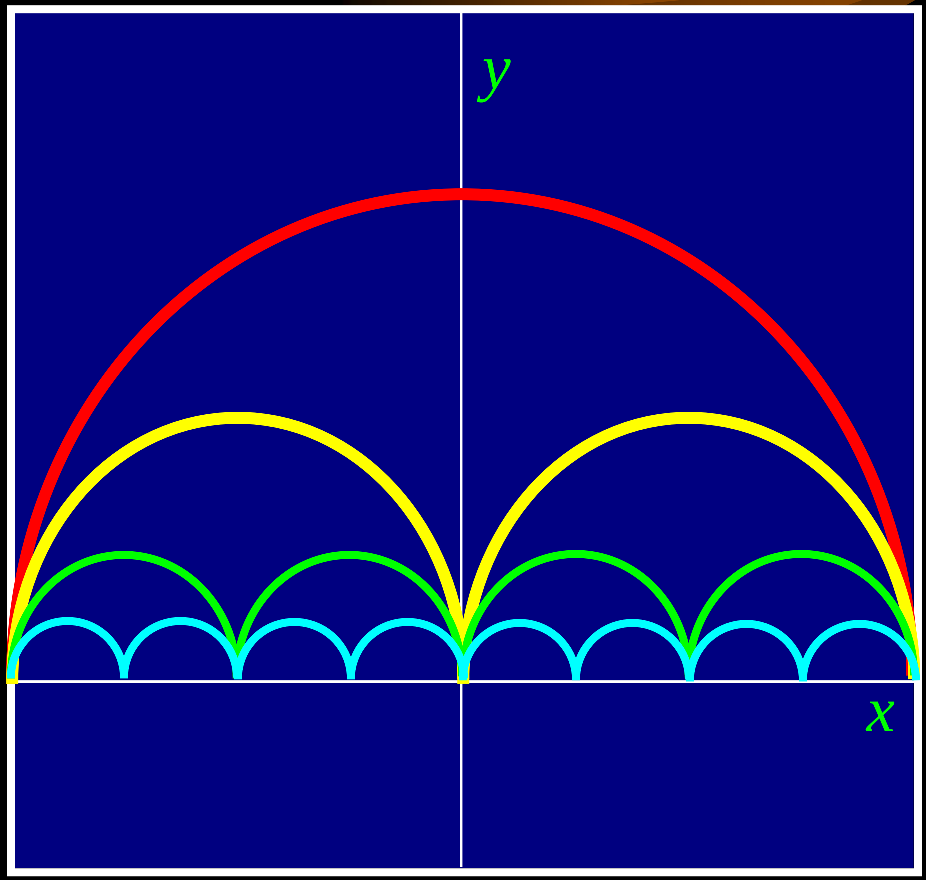
Assume an arc of radius $r = 1$.

Length $= \pi r = \pi$

Length $= 2\,\pi(r/2) = \pi$

Length $= 4\,\pi(r/4) = \pi$

Length $= 8\,\pi(r/8) = \pi$



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Definition 1.29

A function $f(x)$ is called **increasing** on an interval (c,d) if for all $u, v \in (c,d)$, $u \le v$ implies $f(u) \le f(v)$.

Definition 1.30

A function $f(x)$ is called **decreasing** on an interval (c,d) if for all $u, v \in (c,d)$, $u \le v$ implies $f(u) \ge f(v)$.

Theorem 1.31

Suppose $f(x) \in C^{(1)}(c,d)$.

If $f'(x) > 0$ for $x \in (c, d)$, then $f(x)$ is increasing on (c,d).

If $f'(x) < 0$ for $x \in (c, d)$, then $f(x)$ is decreasing on (c,d).

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Definition 1.32

A **(local) maximum** for a function $f \in C^0$ occurs at a point x_0 if there exists $\varepsilon > 0$ so that $f(x_0) \geq f(x)$ for all $x \neq x_0$ such that $| x - x_0 | < \varepsilon$.

A **(local) minimum** to f is defined analogously.

Definition 1.33

The **extermal points** of a function are the points of the graph at which **maxima** or **minima** occur.

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Lemma 1.34

Suppose a function f is piecewise $C^{(1)}$. The extremal points of a function f might occur for only the following values of x:

- $x = a$ and $x = b$, that is the interval endpoints,
- values of x for which $f'(x) = 0$,
- values of x for which $f'(x)$ does not exist.

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Definition 1.37

A function $f(x)$ is called **convex** on (c,d) if for all

$$u,v \in [c,d], \;\; f\left(\frac{u+v}{2}\right) \le \frac{f(u)+f(v)}{2}.$$

Definition 1.38

A function $f(x)$ is called **concave** on (c,d) if for all

$$u,v \in (c,d), \;\; f\left(\frac{u+v}{2}\right) \ge \frac{f(u)+f(v)}{2}.$$

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Theorem 1.39

Suppose $f \in C^2$.

If $f''(x) > 0$ for $x \in (c,d)$, then $f(x)$ is **convex** on (c,d).

If $f''(x) < 0$ for $x \in (c,d)$, then $f(x)$ is **concave** on (c,d).

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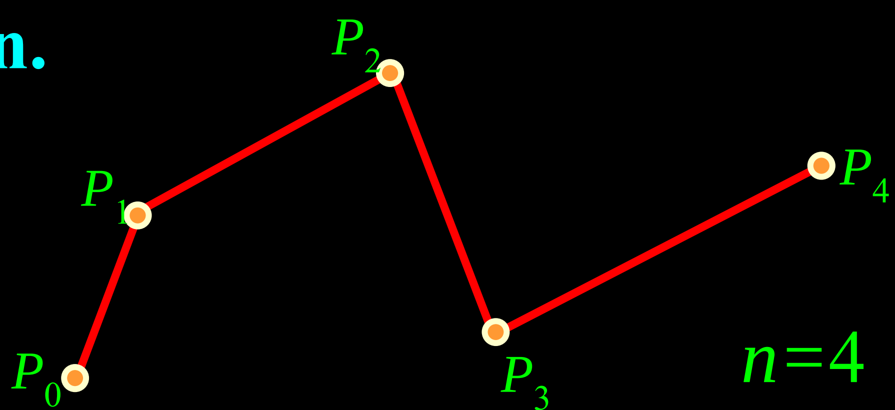
Constructive Bezier Curve Algorithm

Consider the $n+1$ points $P_0,\ldots,P_n$ and connect the points into a polyline that we will denote hereafter as the **control polygon.**



Given points P_i, $i = 0,\ldots,n$, our goal is to determine a curve $\gamma(t)$, for all values $t \in [0,1]$.

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Algorithm 5.1
(Constructive Bezier Curve Algorithm)

Step 1: Select a value $t \in [0,1]$. This value remains constant for the rest of the steps.

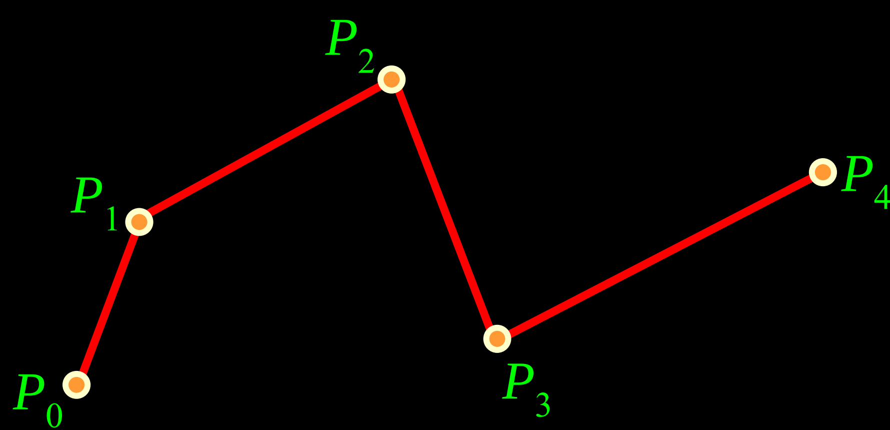
Step 2: Set $P_i^{[0]}(t) = P_i$, for $i = 0,\ldots,n$.

Step 3: For $j = 0,\ldots,n$, set $P_i^{[j]}(t) = (1-t)P_{i-1}^{[j-1]}(t) + tP_i^{[j-1]}(t)$, for $i = j,\ldots,n$.

Step 4: $\gamma(t) = P_n^{[n]}(t)$

$n = 4$

$t = ½$



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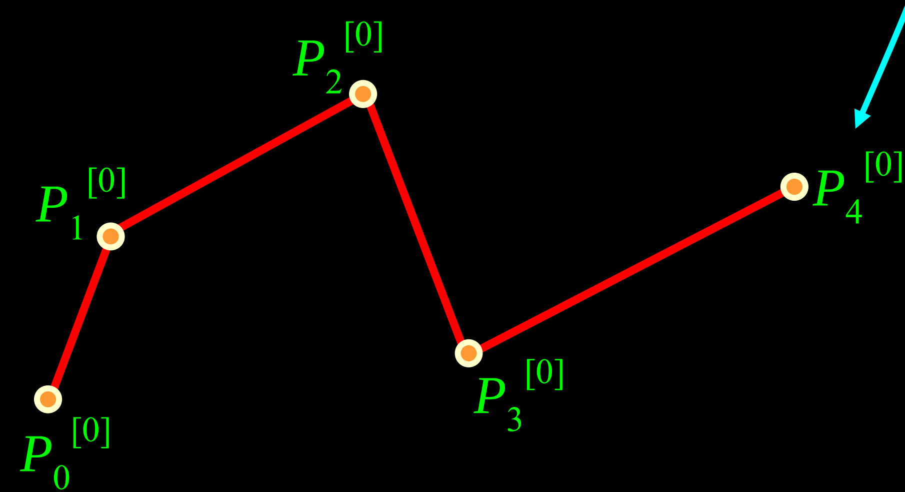

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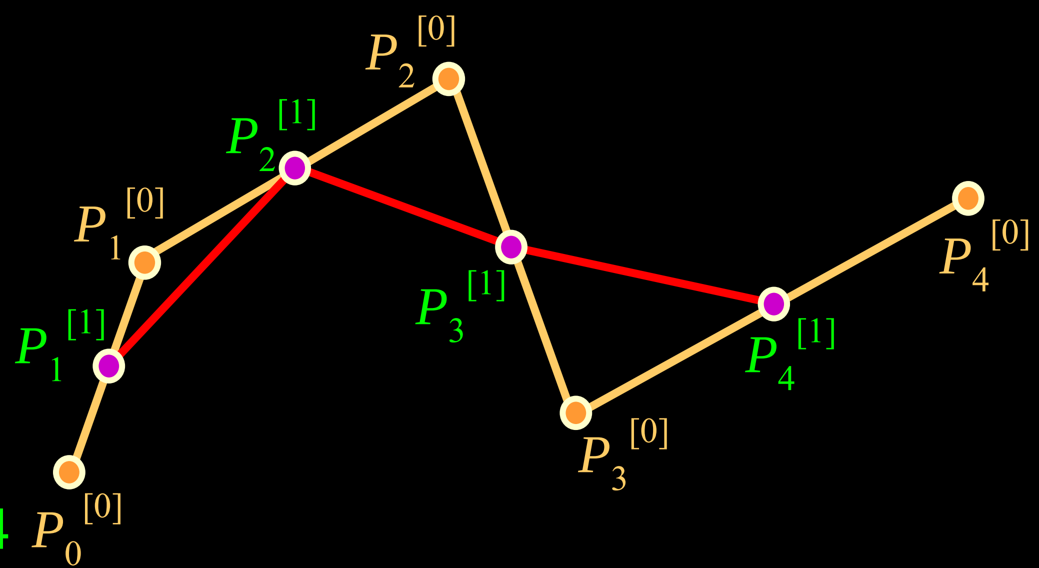
Step 4: $\gamma(t) = P_n^{[n]}(t)$

$n = 4$

$t = ½$

$j = 1$

$$P_i^{[1]} = \frac{1}{2}P_{i-1}^{[0]} + \frac{1}{2}P_i^{[0]},\ i = 1,\ldots,4$$



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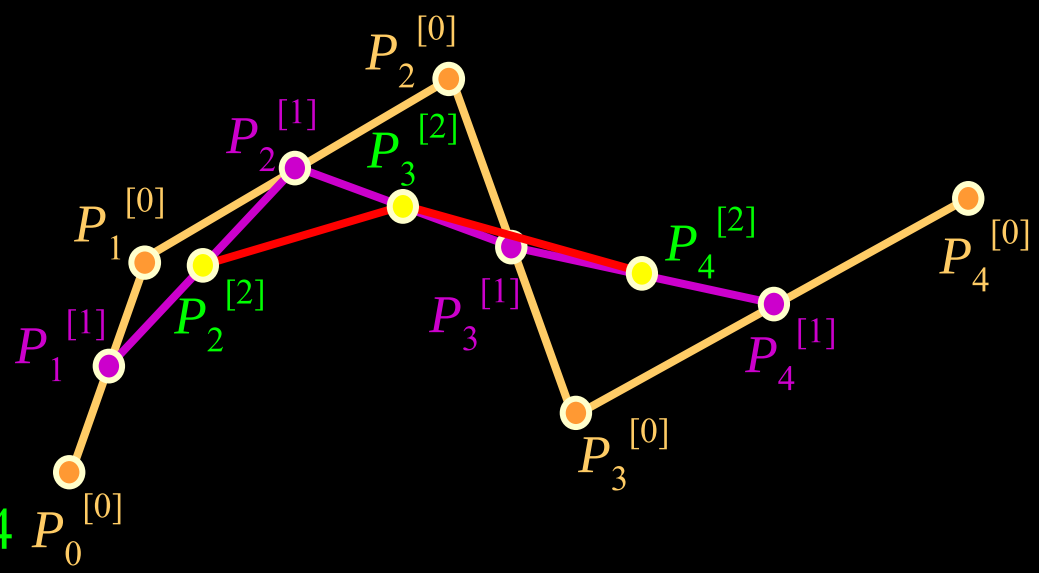
Step 4: $\gamma(t) = P_n^{[n]}(t)$

$n = 4$

$t = ½$

$j = 2$

$$P_i^{[2]} = \frac{1}{2}P_{i-1}^{[1]} + \frac{1}{2}P_i^{[1]},\ i = 2,\ldots,4$$



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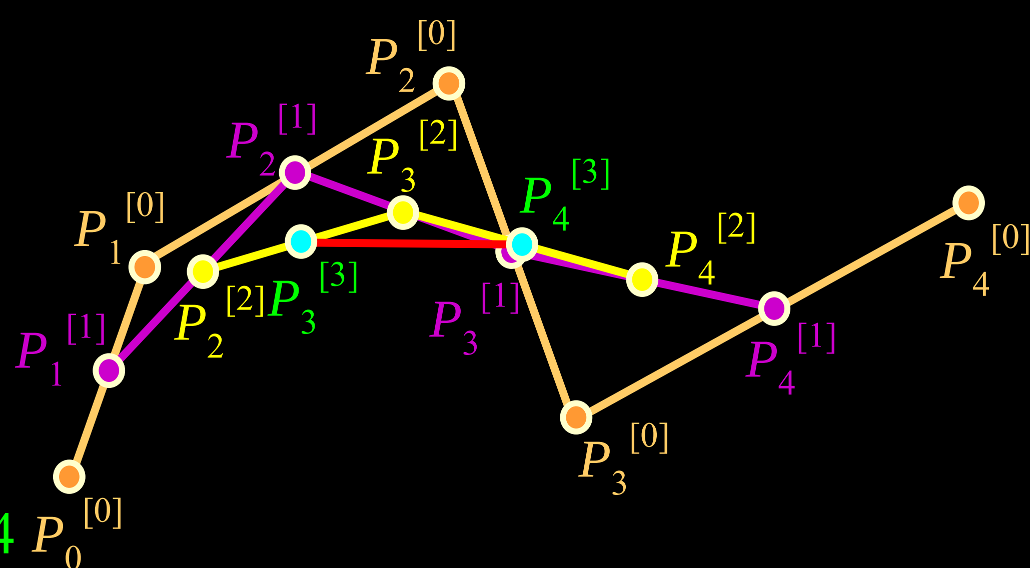
Step 4: $\gamma(t) = P_n^{[n]}(t)$

$n = 4$

$t = ½$

$j = 3$

$$P_i^{[3]} = \frac{1}{2}P_{i-1}^{[2]} + \frac{1}{2}P_i^{[2]}, \; i = 3,\ldots,4$$



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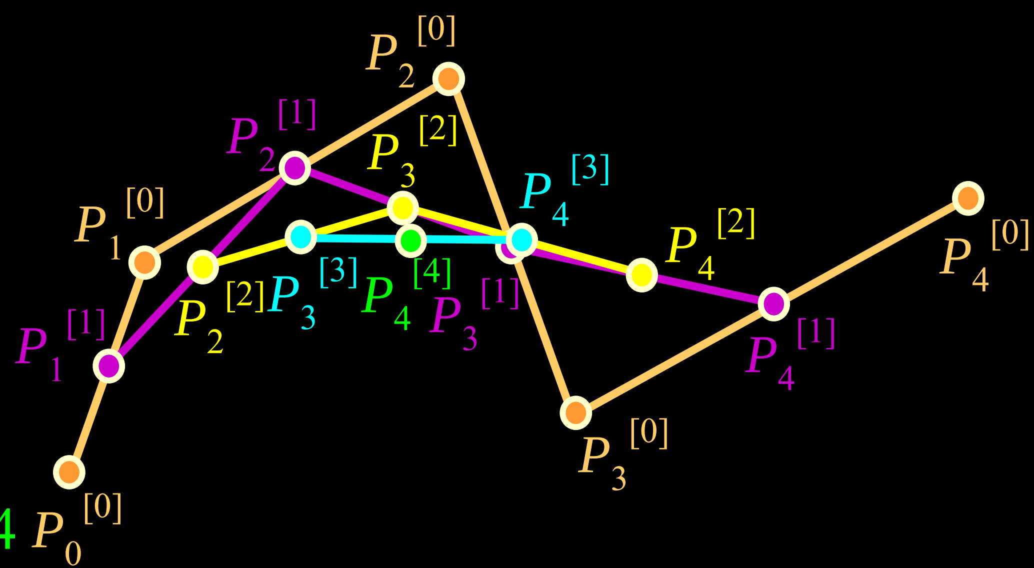
Step 4: $\gamma(t) = P_n^{[n]}(t)$

$n = 4$

$t = ½$

$j = 4$

$$P_i^{[4]} = \frac{1}{2}P_{i-1}^{[3]} + \frac{1}{2}P_i^{[3]}, \; i = 4,\ldots,4$$



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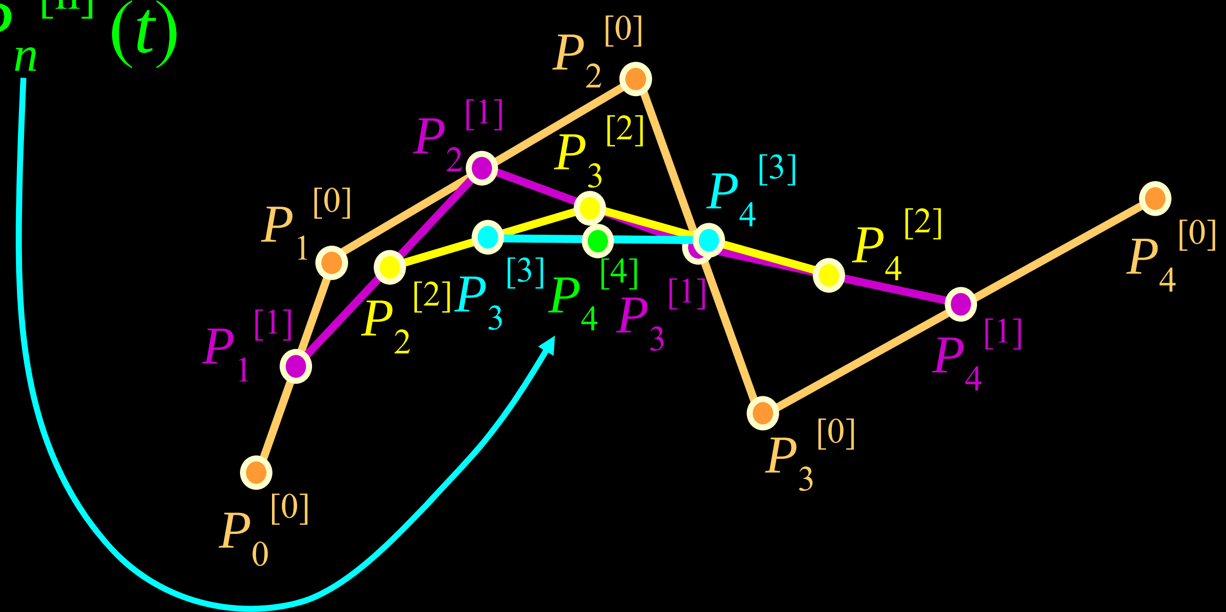

Properties of Bezier Curve $\gamma(t)$

- $\gamma(t)$ is in the convex hull of P_i, $i = 0,\ldots,n$.
- $\gamma(0) = P_0$ and $\gamma(1) = P_n$.
- $\gamma(t)$ looks tangent to the first segment at $t = 0$ and tangent to the last segment at $t = 1$.
- γ is a variation diminishing curve of the control polygon.
- γ seems to follow the general shape of the control polygon.
- (Re)moving P_i, the new shape is easily anticipated.

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Different View of Algorithm 5.1

$$\begin{bmatrix}(1-t) & t\end{bmatrix}\begin{bmatrix}P_0 & P_1 & \cdots & P_{n-1}\\ P_1 & P_2 & \cdots & P_n\end{bmatrix} = \begin{bmatrix}P_1^{[1]} & P_2^{[1]} & \cdots & P_n^{[1]}\end{bmatrix}$$

$$\begin{bmatrix}(1-t) & t\end{bmatrix}\begin{bmatrix}P_1^{[1]} & P_2^{[1]} & \cdots & P_{n-1}^{[1]}\\ P_2^{[1]} & P_3^{[1]} & \cdots & P_n^{[1]}\end{bmatrix} = \begin{bmatrix}P_2^{[2]} & P_3^{[2]} & \cdots & P_n^{[2]}\end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix}(1-t) & t\end{bmatrix}\begin{bmatrix}P_{n-1}^{[n-1]}\\ P_n^{[n-1]}\end{bmatrix} = P_n^{[n]}.$$

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Theorem 5.3

If $\{P_i\}$ is a control polygon, then

$$\gamma(t) = \sum_{i=0}^{n} P_i \frac{n!}{i!(n-i)!}(1-t)^{n-i} t^i,$$

where, as before, $\gamma(t)$ denotes the point on the Bezier curve constructed with Algorithm 5.1 with parameter value t.

Some Conventions

We use the following conventions:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \text{and} \quad \theta_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i,$$

or Theorem 5.3 states that $\gamma(t) = \sum_{i=0}^{n} P_i \theta_{i,n}(t).$

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Theorem 5.4

Before proving Theorem 5.3, we need to prove

$$\theta_{k,n}(t) = \begin{cases} 1, & \text{for } n = 0, \\ t\theta_{k-1,n-1}(t) + (1-t)\theta_{k,n-1}(t), & \text{for } 0 < k < n, \\ (1-t)\theta_{0,n-1}(t), & \text{for } n > 0, k = 0, \\ t\theta_{n-1,n-1}(t), & \text{for } n = k > 0. \end{cases}$$

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Theorem 5.4 (Proof)

We prove only the general case, while the end conditions are left as an exercise.

Because $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we have

$$\binom{n}{k} = \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n}{k}\binom{n-1}{k-1}$$

$$= \frac{n}{n-k}\frac{(n-1)!}{k!((n-1)-k)!} = \frac{n}{n-k}\binom{n-1}{k}.$$

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Theorem 5.4 (Proof Cont.)

For $0 < k < n$, we have

$$\theta_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

$$= \frac{n}{k} \binom{n-1}{k-1} t^k (1-t)^{n-k}$$

$$= \frac{n}{k} t \binom{n-1}{k-1} t^{k-1} (1-t)^{(n-1)-(k-1)}$$

$$= \frac{n}{k} t \theta_{k-1,n-1}(t) \qquad \text{or} \qquad \frac{k}{n} \theta_{k,n}(t) = t \theta_{k-1,n-1}(t).$$

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Theorem 5.4 (Proof Cont.)

Similarly, for $0 < k < n$, we have

$$\theta_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

$$= \frac{n}{n-k}\binom{n-1}{k} t^k (1-t)^{n-k}$$

$$= \frac{n}{n-k}(1-t)\binom{n-1}{k} t^k (1-t)^{(n-1)-k}$$

$$= \frac{n}{n-k}(1-t)\theta_{k,n-1}(t) \text{ or } \quad \frac{n-k}{n}\theta_{k,n}(t) = (1-t)\theta_{k,n-1}(t).$$

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Theorem 5.4 (Proof Cont.)

From $\frac{k}{n}\theta_{k,n}(t) = t\theta_{k-1,n-1}(t)$

and from $\frac{n-k}{n}\theta_{k,n}(t) = (1-t)\theta_{k,n-1}(t)$

we have:

$$\theta_{k,n}(t) = \frac{k}{n}\theta_{k,n}(t) + \frac{n-k}{n}\theta_{k,n}(t)$$
$$= t\theta_{k-1,n-1}(t) + (1-t)\theta_{k,n-1}(t).$$

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Theorem 5.3 (Proof)

Going back to Theorem 5.3, we denote by $\alpha(t)$ the right side of the result we seek,

$$\alpha(t) \overset{def}{=} \sum_{i=0}^{n} P_i \theta_{i,n}(t).$$

By Theorem 5.4 we can write,

$$\begin{aligned}
\alpha(t) &= \sum_{i=0}^{n} P_i \theta_{i,n}(t) \\
&= P_0 \theta_{0,n}(t) + \sum_{i=1}^{n-1} P_i \theta_{i,n}(t) + P_n \theta_{n,n}(t) \\
&= P_0 (1-t) \theta_{0,n-1}(t) + \sum_{i=1}^{n-1} P_i \left[t \theta_{i-1,n-1}(t) + (1-t) \theta_{i,n-1}(t) \right] + P_n t \theta_{n-1,n-1}(t)
\end{aligned}$$

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$$\alpha(t) = P_0(1-t)\theta_{0,n-1}(t) + \sum_{i=1}^{n-1} P_i\left[t\theta_{i-1,n-1}(t) + (1-t)\theta_{i,n-1}(t)\right] + P_n t\theta_{n-1,n-1}(t)$$

Theorem 5.3 (Proof Cont.)

Regrouping,

$$\alpha(t) = (1-t)\sum_{i=0}^{n-1} P_i\theta_{i,n-1}(t) + t\sum_{i=1}^{n} P_i\theta_{i-1,n-1}(t)$$

$$= (1-t)\sum_{i=0}^{n-1} P_i\theta_{i,n-1}(t) + t\sum_{i=0}^{n-1} P_{i+1}\theta_{i,n-1}(t)$$

$$= \sum_{i=0}^{n-1}\left[(1-t)P_i + tP_{i+1}\right]\theta_{i,n-1}(t)$$

$$= \sum_{i=0}^{n-1} P_{i+1}^{[1]}\theta_{i,n-1}(t)$$

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$$\alpha(t) = \sum_{i=0}^{n-1} P_{i+1}^{[1]} \theta_{i,n-1}(t)$$

Theorem 5.3 (Proof Cont.)

Repeating the application of this procedure j times,

$$\alpha(t) = \sum_{i=0}^{n-j} P_{j+i}^{[j]} \theta_{i,n-j}(t),$$

and after n iterations, we will end up with

$$\alpha(t) = P_n^{[n]} \theta_{0,0}(t) = P_n^{[n]} = \gamma(t).$$

Thereby concluding the proof.

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Definition 5.6

The functions

$$\theta_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i,$$

for $i = 0,\ldots,n$, for any nonnegative integer, n, are called the Bernstein basis functions or the Bernstein blending functions.

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Definition 5.7

For $t \in [a,b]$, the functions

$$\theta_{i,n}[t;a,b] = \theta_{i,n}\left(\frac{t-a}{b-a}\right)$$

$$= \binom{n}{i} \frac{(b-t)^{n-i}(t-a)^{i}}{(b-a)^{n}}$$

for $i = 0,\ldots,n$, for any nonnegative integer, n, are called the generalized Bernstein basis functions or the generalized Bernstein blending functions.

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Properties of Bernstein Basis Functions

Theorem 5.8

$\theta_{i,n}(t) > 0$ for $t \in (0,1)$, for all n and $i = 0,\ldots,n$.

Proof

For $t \in (0,1)$, $t > 0$ and $(1- t) > 0$. $\theta_{i,n}(\mathrm{t})$ is then just the product of n positive factors, and is hence positive.

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Theorem 5.9

$$\sum_{i=0}^{n} \theta_{i,n}(t) \equiv 1, \quad \text{for} \quad t \in [0,1].$$

Proof

This follows from the binomial theorem in which

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

Letting $a = (1- t)$ and $b = t$ we have

$$((1-t)+t)^n = 1^n = 1 = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i = \sum_{i=0}^{n} \theta_{i,n}(t).$$

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Corollary

The Bezier curve lies in the convex hull of the points $\{P_i\}_{i=0}^{n}$.

Proof

This follows immediately from Theorems 5.8 and 5.9 (convex combination).

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Theorem

The Bezier curve, that is considered as an approximation to its control polygon, is **variation diminishing**.

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Theorem 5.10

The Bernstein blending functions are **unimodal** with the maximum for the *i*th blending function of degree n at $t = i/n$.

Comment: A function is considered **unimodal** if it has one extremal point over the domain.

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Theorem 5.10 (proof)

Consider $\theta_{0,n}(t) = (1-t)^n$ and $\theta_{n,n}(t) = t^n$. These functions are monotone on the interval $(0,1)$ and hence have one maximum point, at the endpoints.

Now consider that $\theta_{i,n}(t)$ for $i = 1,\ldots,n-1$. $\theta_{i,n}(t)$ is continuous and the maximum of $\theta_{i,n}(t)$ can occur at $t = 0$, $t = 1$ or at $\theta_{i,n}`(t) = 0$.

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Theorem 5.10 (Proof Cont.)

Consider the derivatives, $\theta_{i,n}`(t)$,

$$\theta_{i,n}^{'}(t) = \binom{n}{i}\left[it^{i-1}(1-t)^{n-i} - (n-i)t^{i}(1-t)^{n-1-i}\right]$$

$$= \binom{n}{i}t^{i-1}(1-t)^{n-1-i}\left[i(1-t)-(n-i)t\right]$$

$$= \binom{n}{i}t^{i-1}(1-t)^{n-1-i}\left[i-nt\right]$$

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Theorem 5.10 (Proof Cont.)

The zeros of the derivatives, $\theta_{i,n}`(t) = 0$, in the domain of $(0, 1)$ equal,

$$\theta'_{i,n}(t) = 0 = \binom{n}{i} t^{i-1}(1-t)^{n-1-i}\left[i - nt\right]$$

or

$$t_e = 0 \quad \text{or} \quad t_e = 1 \quad \text{or} \quad t_e = \frac{i}{n}.$$

Because $\theta_{i,n}(0) = \theta_{i,n}(1) = 0$ while being positive in $(0, 1)$, only one maximum can exist at $t = i/n$.

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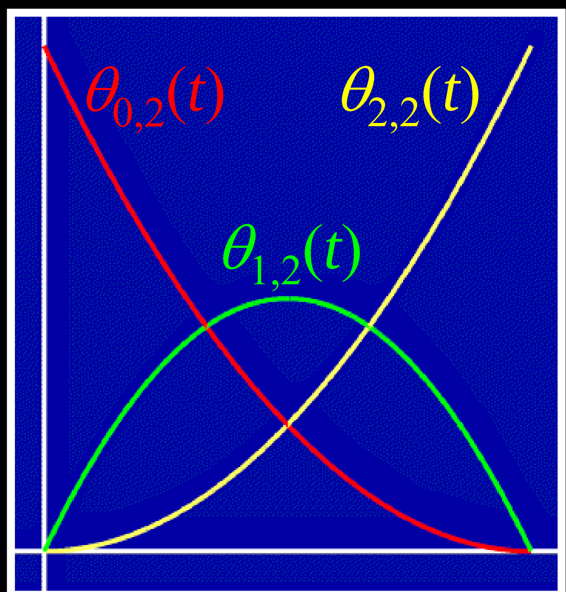
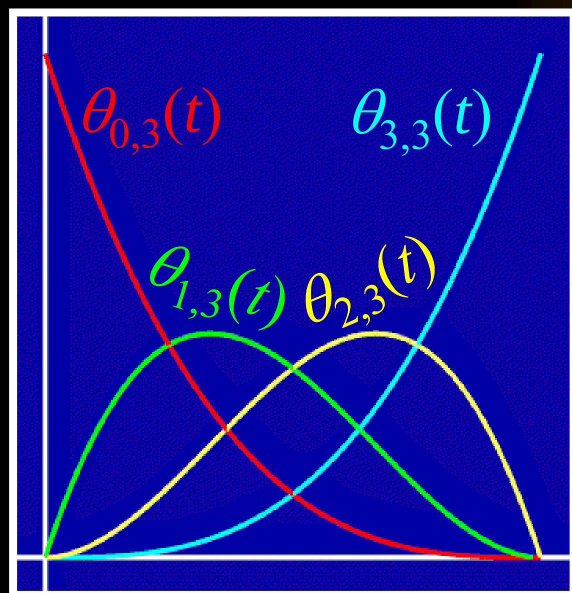
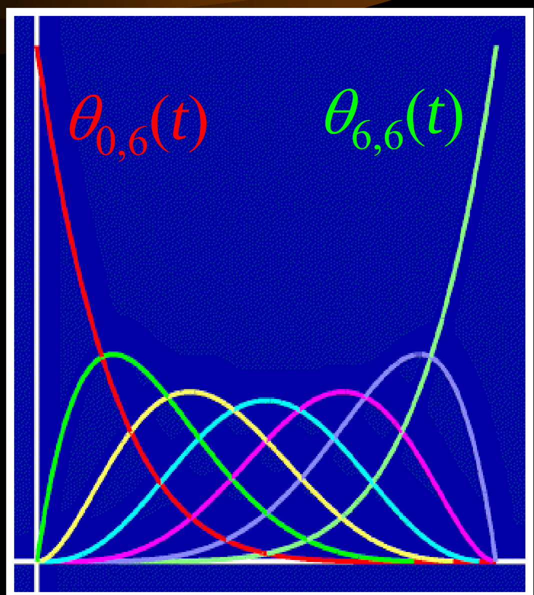
Examples of Basis Functions



Quadratic



Cubic



Degree 6

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Theorem 5.12

The Bernstein blending functions of degree n form a basis for the polynomials of degree n.

Proof: $$0 = \sum_{i=0}^{n} c_i \theta_{i,n}(t)$$

$$= \sum_{i=0}^{n} c_i \binom{n}{i} t^i (1-t)^{n-i}$$

$$= \sum_{i=0}^{n} c_i \binom{n}{i} t^i \sum_{j=0}^{n-i} \binom{n-i}{j} (-t)^{n-i-j}$$

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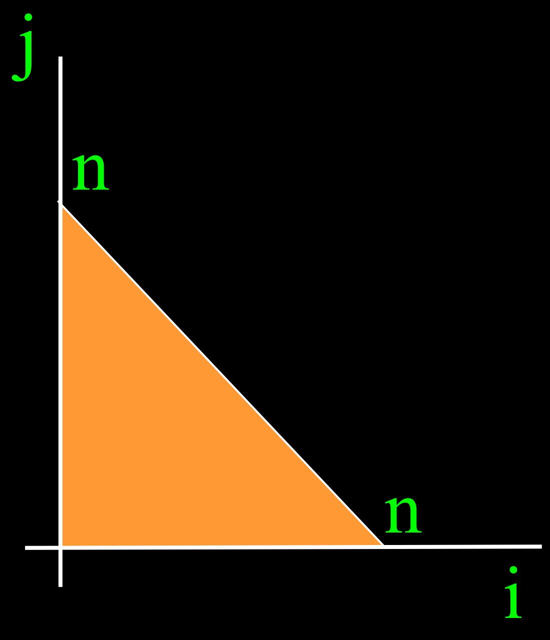
Theorem 5.12 (Cont.)

Continuing,



$$0 = \sum_{i=0}^{n} c_i \binom{n}{i} t^i \sum_{j=0}^{n-i} \binom{n-i}{j} (-t)^{n-i-j}$$

$$= \sum_{i=0}^{n} c_i \binom{n}{i} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} t^{n-j}$$

$$= \sum_{j=0}^{n} t^{n-j} \sum_{i=0}^{n-j} c_i \underbrace{(-1)^{n-i-j} \binom{n}{i}\binom{n-i}{j}}_{d_i \neq 0}$$

$$= \sum_{j=0}^{n} t^{n-j} \sum_{i=0}^{n-j} c_i d_i .$$

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Theorem 5.12 (Cont.)

Consider $$0=\sum_{j=0}^{n} t^{n-j}\sum_{i=0}^{n-j} c_i d_i, \quad d_i \neq 0.$$

For $j=n$, it falls immediately that c_0 equals zero.

For $j=n-1$, it falls immediately that c_1 equals zero.

In a similar fashion, it is simple to show that all c_i , $i=0,\ldots n$, are zero.

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Derivative Evaluation

Differentiating $\theta_{i,n}(t)$, one gets:

$$\theta'_{i,n}(t) = \binom{n}{i}\left[it^{i-1}(1-t)^{n-i} - (n-i)t^i(1-t)^{n-i-1}\right].$$

Now $$\binom{n}{i}i = \frac{n!i}{i!(n-i)!} = n\frac{(n-1)!}{(i-1)!(n-i)!} = n\binom{n-1}{i-1},$$

and $$\binom{n}{i}(n-i) = \frac{n!(n-i)}{i!(n-i)!} = n\frac{(n-1)!}{i!(n-1-i)!} = n\binom{n-1}{i},$$

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Derivative Evaluation (Cont.)

or $$\theta'_{i,n}(t) = n\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i} - n\binom{n-1}{i}t^{i}(1-t)^{n-i-1}$$

$$= n\left(\theta_{i-1,n-1}(t) - \theta_{i,n-1}(t)\right).$$

For i=0, $\theta_{0,n}(t)=(1-t)^n$ and we have

$$\theta'_{0,n}(t) = \binom{n}{0}(-n)(1-t)^{n-1} = -n\binom{n-1}{0}(1-t)^{n-1} = -n\theta_{0,n-1}(t),$$

and similarly for i=n.

Theorem 5.15

When $\theta_{i,n}(t)$ denotes the ith Bernstein blending function of degree n, then

$$\theta'_{i,n}(t) = n\left(\theta_{i-1,n-1}(t) - \theta_{i,n-1}(t)\right) \qquad \text{for} \quad i = 0,\ldots,n,$$

where

$$\theta_{-1,n-1}(t) \equiv 0 \quad \text{and} \quad \theta_{n,n-1}(t) \equiv 0.$$

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Corollary 5.16

If $\gamma(t) = \sum_{i=0}^{n} P_i \theta_{i,n}(t)$, then

$$\gamma'(t) = \sum_{i=0}^{n-1} Q_i \theta_{i,n-1}(t),$$

where

$$Q_i = n(P_{i+1} - P_i).$$

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Corollary 5.17

If $\gamma(t) = \sum_{i=0}^{n} P_i \theta_{i,n}(t)$, then

$$\gamma^{(j)}(t) = \sum_{i=0}^{n-j} Q_i^{[j]} \theta_{i,n-j}(t),$$

where

$$Q_i^{[j+1]} = \begin{cases} P_i, & j = -1, \\ (n-j)\left[Q_{i+1}^{[j]} - Q_i^{[j]}\right], & j = 0, \ldots, n-1. \end{cases}$$

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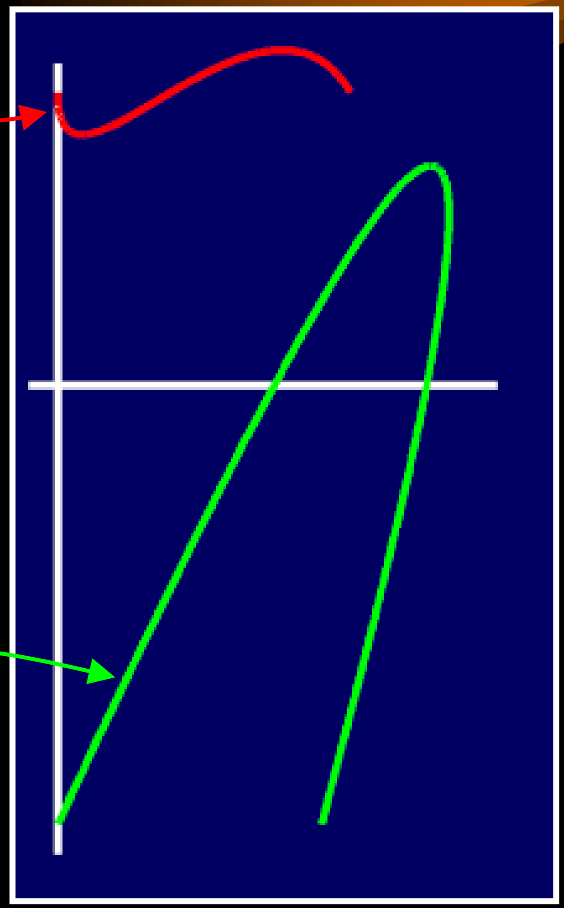
The Hodograph

Given a **curve**, the graph of its derivative curve is also denoted the **Hodograph curve**.



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Interpolation Using the Bernstein Blending Functions

Question: Can we employ the Bezier/Bernstein polynomials for interpolation?

Question: Can we coerce Bezier curve $\gamma(t) = \Sigma P_i \theta_i(t)$ to interpolate certain locations? For example, $\gamma(i/n) = V_i$, $i=0,\ldots,n$?

Points i/n, $i=0,\ldots,n$ are known as node points.

Question: What is unique about the node points?

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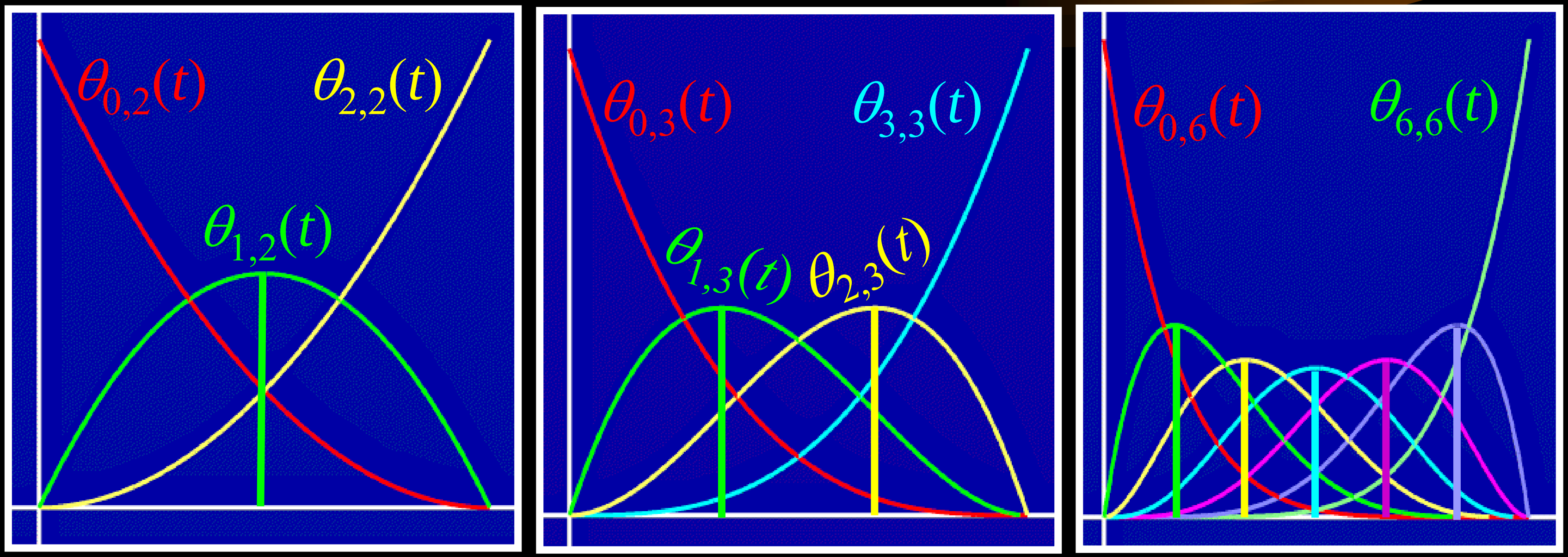
Nodal Points



Quadratic Cubic Degree 6

Interpolation Using the Bernstein Blending Functions

Impose the interpolation problem as a linear system of equations, **$\Theta P = V$**:

$$\left[\theta_{j,n}(i/n)\right]\left[P_0 \quad P_1 \quad \ldots \quad P_n\right]^T = \left[V_0 \quad V_1 \quad \ldots \quad V_n\right]^T$$

or,

$$\begin{bmatrix} \theta_{0,n}\left(\frac{0}{n}\right) & \theta_{1,n}\left(\frac{0}{n}\right) & \theta_{2,n}\left(\frac{0}{n}\right) & & \theta_{n,n}\left(\frac{0}{n}\right) \\ \theta_{0,n}\left(\frac{1}{n}\right) & \theta_{1,n}\left(\frac{1}{n}\right) & \theta_{2,n}\left(\frac{1}{n}\right) & & \theta_{n,n}\left(\frac{1}{n}\right) \\ \theta_{0,n}\left(\frac{2}{n}\right) & \theta_{1,n}\left(\frac{2}{n}\right) & \theta_{2,n}\left(\frac{2}{n}\right) & & \theta_{n,n}\left(\frac{2}{n}\right) \\ & & & & \\ \theta_{0,n}\left(\frac{n}{n}\right) & \theta_{1,n}\left(\frac{n}{n}\right) & \theta_{2,n}\left(\frac{n}{n}\right) & & \theta_{n,n}\left(\frac{n}{n}\right) \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}.$$

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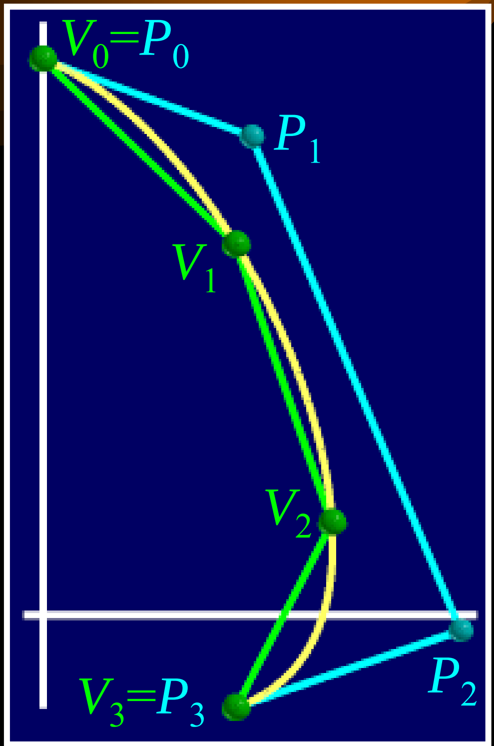
Interpolation Using the Bernstein Blending Functions (Example 5.30)

Consider a cubic Bezier interpolating the following four points, $V_0=(0,6)$, $V_1=(2,4)$, $V_2=(3,1)$, $V_3=(2,-1)$:



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 8/27 & 4/9 & 2/9 & 1/27 \\ 1/27 & 2/9 & 4/9 & 8/27 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} (0,6) \\ (2,4) \\ (3,1) \\ (2,-1) \end{bmatrix}$$

Question: What can one say about the solution’s existence? About the stability (condition number) of Θ?

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Bernstein Approximation

Question: By sampling curve $f(x)$, $x \in [0,1]$ at the node points $f(i/n)=V_i$, $i=0,\ldots,n$, can a Bezier curve reproduce

- Constant $f(x)$ functions?
- Linear $f(x)$ functions?
- Quadratic $f(x)$ function?
- Arbitrary order $f(x)$ functions?

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Definition 5.19

For a function $f \in C^{(m)}[0,1]$, the nth Bernstein polynomial approximation is

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\theta_{k,n}(x).$$

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Definition 5.20

A sequence of functions $\{s_j(x)\}_j$ is said to converge uniformly to a function $s(x)$ on an interval I if for every $\varepsilon > 0$ there exists a natural number N such that, for all $j > N$, $| s_j(x) - s(x) | < \varepsilon$.

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Theorem 5.21

If $f(x)$ is bounded on $[0,1]$, then

$$\lim_{n\to\infty} B_n(f;x) = f(x)$$

at any point x where f is continuous. If $f \in C^{(m)}[0,1]$, then it converges uniformly. Further, if $f \in C^{(m)}[0,1]$, then $\lim_{n\to\infty} B_n^{(m)}(f;x) = f^{(m)}(x)$ uniformly on $[0,1]$.

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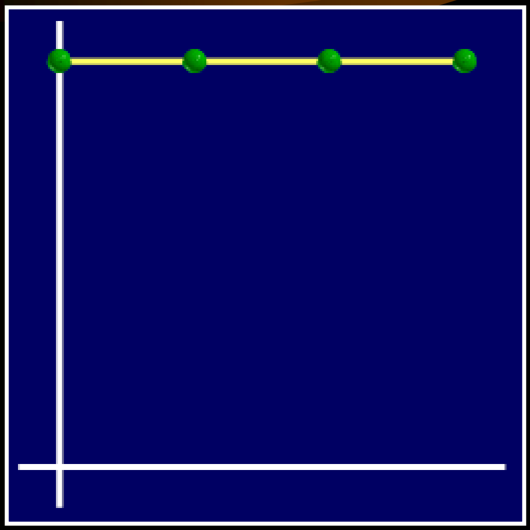
Example 5.22 ($f(x) = 1$ Function)

If $f(x)$ is constant one, we have,



$$B_n(1;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$
$$= \left(x + (1-x)\right)^n$$
$$= 1.$$

In other words, the Bernstein approximation reproduces constants.

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Example 5.23 (Linear $f(x)$)

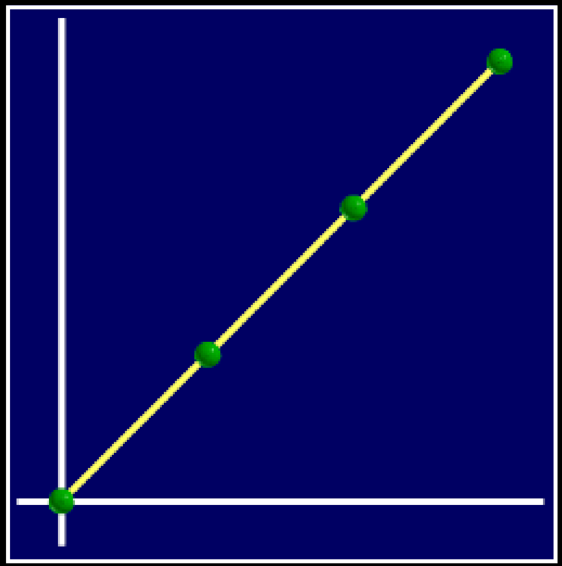
If $f(x) = x$, we have $f(k/n) = k/n$ and,

$$B_n(x;x) = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= x\sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)}$$



reproducing linear functions. $= x\sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{(n-1)-k} = x.$

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Example 5.24 (Quadratic $f(x)$)

If $f(x)$ is quadratic, assume $f(x) = x^2$ and we have,

$$B_n(x^2; x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

In contrast, assume $n \geq 2$,

$$x^2 = x^2 \sum_{r=0}^{n-2} \binom{n-2}{r} x^r (1-x)^{n-2-r} = \sum_{r=0}^{n-2} \binom{n-2}{r} x^{r+2} (1-x)^{n-2-r}$$

which yields,

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Quadratic *f*(*x*) (Cont.)

$$x^2 = \sum_{r=0}^{n-2} \binom{n-2}{r} x^{r+2}(1-x)^{n-2-r}$$

$$= \sum_{r=0}^{n-2} \frac{(r+2)(r+1)}{n(n-1)} \binom{n}{r+2} x^{r+2}(1-x)^{n-(r+2)}$$

$$= \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^{k}(1-x)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^{k}(1-x)^{n-k}$$

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Quadratic $f(x)$ (Cont.)

So we have,

$$B_n(x^2;x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k},$$

and $x^2 = \sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^k (1-x)^{n-k},$

and it is clear that $B_n(x^2;x) \neq x^2$.

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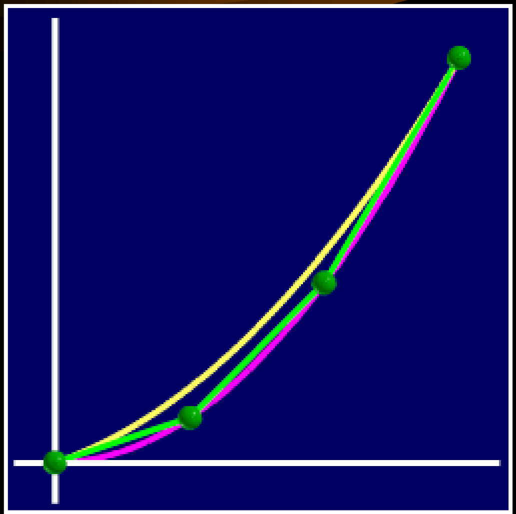
Quadratic *f*(*x*) (Cont.)

The error (difference) equals,



$$\left|B_n(x^2;x)-x^2\right| = \sum_{k=0}^{n}\left|\frac{k^2}{n^2}-\frac{k(k-1)}{n(n-1)}\right|\theta_{k,n}(t)$$

$$= \sum_{k=0}^{n}\left|\frac{k}{n}\left(\frac{k}{n}-\frac{(k-1)}{(n-1)}\right)\right|\theta_{k,n}(t)$$

$$= \sum_{k=0}^{n}\left|\frac{k}{n^2}\left(\frac{n-k}{n-1}\right)\right|\theta_{k,n}(t)$$

$$\le \sum_{k=0}^{n}\frac{1}{n}\theta_{k,n}(t), \quad \text{or} \quad \left|B_n(x^2;x)-x^2\right| \le 1/n.$$

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Lemma 5.26

Suppose $s(x) = ax + b$ is an arbitrary straight line and $B_n(f;x)$ is the nth degree Bernstein approximation. Then, the number of times $B_n(f;x)$ crosses s is exactly the same as the number of sign changes of the Bernstein approximation to $f - s$.

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Proof (of Lemma 5.26)

$$B_n(f-s;x)=\sum_{i=0}^{n}\left(f(i/n)-s(i/n)\right)\theta_{i,n}(x)$$

$$=\sum_{i=0}^{n}f(i/n)\theta_{i,n}(x)-\sum_{i=0}^{n}s(i/n)\theta_{i,n}(x)$$

$$=\sum_{i=0}^{n}f(i/n)\theta_{i,n}(x)-s(x),$$

because $B_n(x;\, x) = x$, and $B_n(1;\, x) = 1$.

Counting zeros is equivalent to counting intersections.

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Descartes' Rule of Signs

If

$$P(x) = \sum_{r=0}^{n} p_r x^r$$

has all real coefficients p_r, then the number N_p of positive zeros ($x > 0$) does not exceed the number of sign changes in the sequence $p_n, p_{n-1}, \ldots, p_0$.

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Variation Diminishing

Denote by $Z(a_0, a_1, \ldots, a_n)$ the number of sign changes of the sequence $a_0, a_1, \ldots, a_n$.

Denote by $Z[f\,]$ the number of sign changes of an arbitrary continuous function f.

Variation Diminishing (Cont.)

Theorem 5.27

For an arbitrary continuous function, f, over the interval $[0,1]$,

$$Z[B_n(f;x)] \leq Z\left(f(0), f(1/n), \ldots, f(n/n)\right).$$

Question: Why is it that Theorem 5.27 also proves the variation diminishing property of the Bezier curves?

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Variation Diminishing (Cont.)

Proof (of Theorem 5.27)

For $x \in (0, 1)$,

$$Z\left[B_n(f;x)\right] = Z\left[\frac{B_n(f;x)}{(1-x)^n}\right]$$

$$= Z\left[\sum_{i=0}^{n} f(i/n)\binom{n}{i}\left(\frac{x}{1-x}\right)^i\right] = Z\left[\sum_{i=0}^{n} f(i/n)\binom{n}{i} z^i\right],$$

where $z = x/(1-x)$, taking all positive real values.

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Variation Diminishing (Cont.)

Proof (of Theorem 5.27)

However,

$$Z\left[\sum_{i=0}^{n} f(i/n)\binom{n}{i}z^i\right] \le Z\left(f\left(\frac{0}{n}\right)\binom{n}{0}, f\left(\frac{1}{n}\right)\binom{n}{1},\ldots, f\left(\frac{n}{n}\right)\binom{n}{n}\right)$$

$$= Z\left(f\left(\frac{0}{n}\right), f\left(\frac{1}{n}\right),\ldots, f\left(\frac{n}{n}\right)\right)$$

by the Descartes rule of signs.

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Variation Diminishing in R^3

Definition 5.29

If $f(t)$ is a parametric curve in R^3 and $V[f](t)$ is a parametric variation diminishing approximation of $f(t)$, then an arbitrary plane can intersect $V[f](t)$ no more often than it intersects $f(t)$.

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Piecing Together Bezier Curves

We are interested in the question of connecting different Bezier curves with some prescribed continuity G^k

Let $\gamma_1(t) = \sum_{i=0}^{N} P_i \theta_{i,N}(t)$ and $\gamma_2(t) = \sum_{i=0}^{N} Q_i \theta_{i,N}(t)$ be two Bezier curves and denote by $\gamma(t)$ their compound curve,

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [0,1) \\ \gamma_2(t-1), & t \in [1,2]. \end{cases}$$

Question: What should P_i and Q_i be for G^k continuity?

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Piecing Together Bezier Curves (Cont.)

From Chapter 4, we know that the condition for G^1 continuity equals $\gamma'_1(1) = k_1 \gamma'_2(0)$.

For two n'th degree Bezier curves, we have,

$\gamma'_1(1) = n(P_n - P_{n-1})$ and $\gamma'_2(0) = n(Q_1 - Q_0)$,

or $(P_n - P_{n-1}) = k_1(Q_1 - Q_0)$ and recalling $P_n = Q_0$.

Question: What are the conditions for C^1 continuity?

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Piecing Together Bezier Curves (Cont.)

From Chapter 4, we know that the condition for G^2 continuity equals $\gamma''_1(1) = c_3\, \gamma'_2(0) + (k_1)^2 \gamma''_2(0)$.

For two n'th degree Bezier curves, we have,

$\gamma''_1(1) = n\,(n-1)\,(P_n - 2P_{n-1} + P_{n-2})$ and

$\gamma''_2(0) = n\,(n-1)\,(Q_0 - 2Q_1 + Q_2)$.

Having C^0 continuity that yields $P_n = Q_0$, and assuming G^1 continuity or $(P_n - P_{n-1}) = k_1(Q_1 - Q_0)$,

$\gamma''_1(1) = c_3\, \gamma'_2(0) + (k_1)^2 \gamma''_2(0).$

Piecing Together Bezier Curves (Cont.)

$$n(n-1)(P_n - 2P_{n-1} + P_{n-2}) = c_3 n(Q_1 - Q_0) + (k_1)^2 n(n-1)(Q_0 - 2Q_1 + Q_2)$$

or

$$(n-1)(P_n - 2P_{n-1} + P_{n-2}) = c_3 (Q_1 - Q_0) + (k_1)^2 (n-1)(Q_0 - 2Q_1 + Q_2)$$

or

$$(P_{n-2} - 2P_{n-1} + P_n) = (k_1)^2 (Q_0 - 2Q_1 + Q_2) + \frac{c_3}{n-1}(Q_1 - Q_0)$$

By the G^1 continuity, $(P_n - P_{n-1}) = k_1(Q_1 - Q_0)$, and,

$$P_{n-2} - P_{n-1} = \left(k_1 - \frac{c_3}{n-1} + (k_1)^2 \right)(Q_0 - Q_1) - (k_1)^2 (Q_2 - Q_1)$$

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$$P_{n-2} - P_{n-1} = \left(k_1 - \frac{c_3}{n-1} + (k_1)^2\right)(Q_0 - Q_1) - (k_1)^2(Q_2 - Q_1)$$

Piecing Together Bezier Curves (Cont.)

By G^1 continuity, $P_{n-1} = P_n - k_1(Q_1 - Q_0)$, and since $P_n = Q_0$ from C^0 continuity,

$$P_{n-2} = P_{n-1} + \left(k_1 - \frac{c_3}{n-1} + (k_1)^2\right)(Q_0 - Q_1) - (k_1)^2(Q_2 - Q_1)$$

$$= P_n - k_1(Q_1 - Q_0) + \left(k_1 - \frac{c_3}{n-1} + (k_1)^2\right)(Q_0 - Q_1) - (k_1)^2(Q_2 - Q_1)$$

$$= Q_0 + \left(2k_1 - \frac{c_3}{n-1} + (k_1)^2\right)(Q_0 - Q_1) - (k_1)^2(Q_2 - Q_1).$$

Question: What if $\gamma_1(t)$ and $\gamma_2(t)$ have different orders?

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Degree Raising

Question: Given a Bezier curve, $\gamma(t)$, of degree n, can one represent $\gamma(t)$ as a Bezier curve of degree m, $m > n$? $m < n$?

Question: Why would one need a Bezier curve in a different degree of the exact same shape?

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Degree Raising (Cont.)

We recall the following combinatorial identities,

$$\frac{k}{n+1}\theta_{k,n+1} = t\theta_{k-1,n}(t) \quad \text{and} \quad \frac{n+1-k}{n+1}\theta_{k,n+1}(t) = (1-t)\theta_{k,n}(t).$$

Rewriting the first identity as $\frac{k+1}{n+1}\theta_{k+1,n+1}(t) = t\theta_{k,n}(t)$,

we get the sum of

$$\theta_{k,n}(t) = \frac{k+1}{n+1}\theta_{k+1,n+1}(t) + \frac{n+1-k}{n+1}\theta_{k,n+1}(t).$$

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$$\theta_{k,n}(t) = \frac{k+1}{n+1}\theta_{k+1,n+1}(t) + \frac{n+1-k}{n+1}\theta_{k,n+1}(t).$$

Degree Raising (Cont.)

Given a curve, $\gamma_p(t)$, the degree raised curve equals,

$$\begin{aligned}
\gamma_p(t) &= \sum_{i=0}^{n} P_i \theta_{i,n}(t) \\
&= \sum_{i=0}^{n} P_i \left[\frac{i+1}{n+1}\theta_{i+1,n+1}(t) + \frac{n+1-i}{n+1}\theta_{i,n+1}(t) \right] \\
&= \sum_{i=0}^{n+1} \left[\frac{iP_{i-1} + (n+1-i)P_i}{n+1} \right] \theta_{i,n+1}(t),
\end{aligned}$$

with the assumption that P_{-1} and P_{n+1} are “don’t care”.

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Theorem 5.31

A Bezier curve $\gamma(t) = \sum_{i=0}^{n} P_i \theta_{i,n}(t)$ can be represented as

$$\gamma(t) = \sum_{i=0}^{n+1} Q_i \theta_{i,n+1}(t)$$

where

$$Q_i = \frac{iP_{i-1} + (n+1-i)P_i}{n+1}.$$

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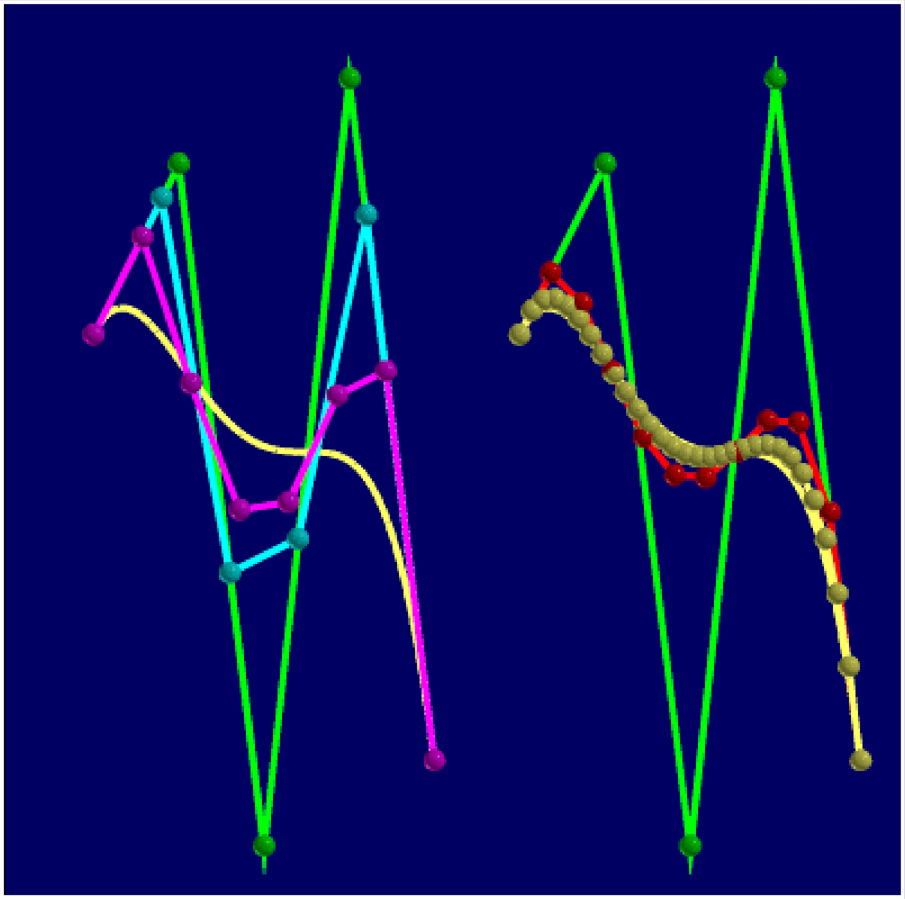
Theorem 5.31 (Cont.)

degree 4

degree 5

degree 7



degree 4

degree 11

degree 19

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Rational Bezier Curves

Let $\theta_{0,n}(t), \ldots, \theta_{n,n}(t)$ be the $n+1$ Bezier basis function of degree n. For a sequence of coefficients $\{P_0, \ldots, P_n\}$, which can be scalar or vector and a sequence of scalars, $\{w_0, \ldots, w_n\}$, the curve

$$\gamma(t) = \frac{\sum_{i=0}^{n} w_i P_i \theta_{i,n}(t)}{\sum_{i=0}^{n} w_i \theta_{i,n}(t)},$$

is called a **rational Bezier curve** of degree n.

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Rational Bezier Curves

The w_i's are sometimes called **homogeneous coordinates** for the P_i's since there is one w_i associated with each P_i . The determining parameters of the rational Bezier curve are the **homogeneous point**

$H_i = (h_{x,i}, h_{y,i}, h_{z,i}, h_{w,i})$ where if $P_i = (x_i, y_i, z_i)$,

$h_{x,i} = w_i x_i$, $h_{y,i} = w_i y_i$, $h_{z,i} = w_i z_i$ and $h_{w,i} = w_i$.

Warning: Some systems use $P_i = (w_i x_i, w_i y_i, w_i z_i)$!

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Integrals of Bezier Basis Functions

Question: How can we compute the integral of a Bezier basis function?

Recall the derivative of a Bezier basis function,

$$\theta'_{i,n}(t) = n\left(\theta_{i-1,n-1}(t) - \theta_{i,n-1}(t)\right).$$

Then,

$$\int \theta_{i-1,n-1}(t)dt = \frac{1}{n}\theta_{i,n}(t) + \int \theta_{i,n-1}(t)dt.$$

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$$\int \theta_{i-1,n-1}(t)dt = \frac{1}{n}\theta_{i,n}(t) + \int \theta_{i,n-1}(t)dt$$

Integrals of Bezier Basis Functions (Cont.)

Or, $\int \theta_{i-1,n-1}(t)dt = \frac{1}{n}\theta_{i,n}(t) + \frac{1}{n}\theta_{i+1,n}(t) + \int \theta_{i+1,n-1}(t)dt$

$$= \frac{1}{n}\sum_{j=i}^{n-1}\theta_{j,n}(t) + \int \theta_{n-1,n-1}(t)dt$$

$$= \frac{1}{n}\sum_{j=i}^{n-1}\theta_{j,n}(t) + \frac{1}{n}\theta_{n,n}(t) = \frac{1}{n}\sum_{j=i}^{n}\theta_{j,n}(t).$$

Question: What is the integral of a Bezier function,

$\int c(t)dt = \int \sum_i P_i\theta_{i,n}(t)dt,$ equal to?

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Products of Bezier (Basis) Functions

Products (and integrals) of geometry are crucial analysis and/or synthesis operators:

- Distances between geometric entities,
- Area/Volume/Moments of geometric entities,
- Curvature analysis, recalling $\kappa(t) = \left\| \beta' \times \beta'' \right\| / \left\| \beta' \right\|^3$.

Question: What about arc-length computation?

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Products of Bezier Basis Functions

Recall that $\theta_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i,$

Then, $\theta_{i,n}(t)\theta_{j,m}(t) = \binom{n}{i}(1-t)^{n-i}t^i\binom{m}{j}(1-t)^{m-j}t^j$

$$= \binom{n}{i}\binom{m}{j}(1-t)^{n+m-i-j}t^{i+j}.$$

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$$\theta_{i,n}(t)\theta_{j,m}(t) = \binom{n}{i}\binom{m}{j}(1-t)^{n+m-i-j}t^{i+j}.$$

Products of Bezier Basis Functions.

But also by definition $\theta_{i+j,n+m}(t) = \binom{n+m}{i+j}(1-t)^{n+m-i-j}t^{i+j}$,

or, $$\theta_{i,n}(t)\theta_{j,m}(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}}\theta_{i+j,n+m}(t).$$

Question: To what is the product of two Bezier functions, $c_1(t)\ c_2(t)$, equal?

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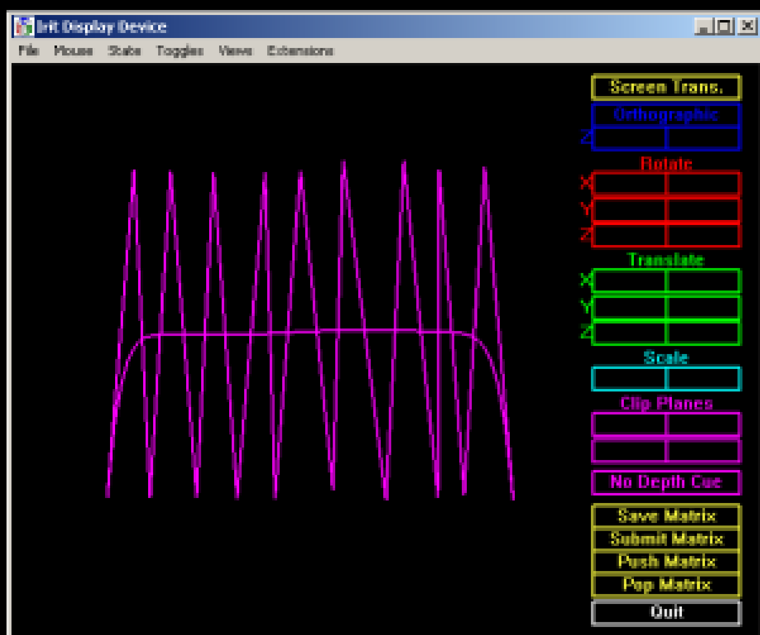
Bezier Demo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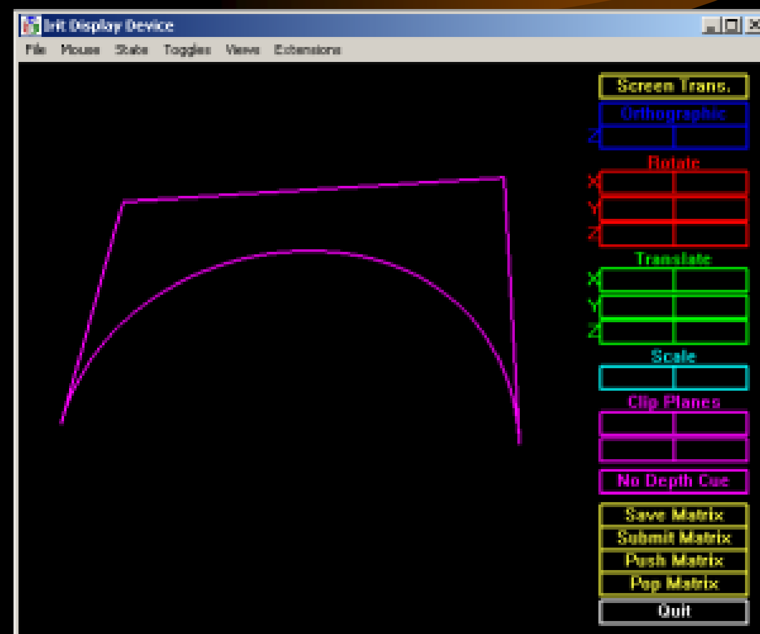
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