Computer Aided Geometric Design

B-spline Curves

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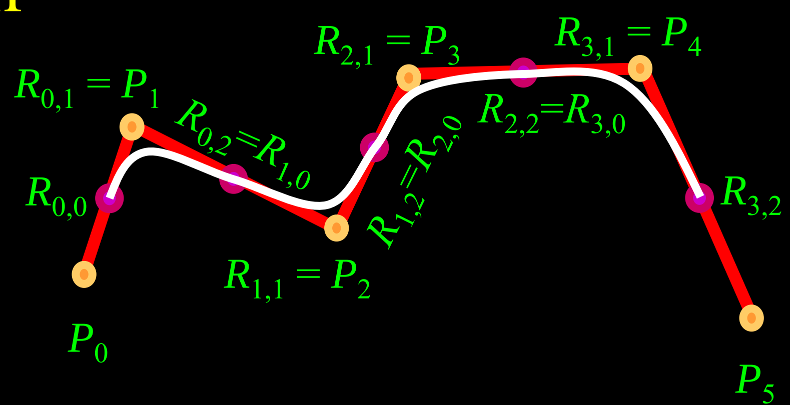
based on a book by Cohen, Riesenfeld, & Elber

Constructive Piecewise Curves

Consider an arbitrary polygon, $\{ P_i \}_{i=0}^{n}$.

Let $\boldsymbol{u} = \{ u_i \}_{i=0}^{n}$ denote a collection of distinct values over which a piecewise quadratic Bezier function will be defined. The j'th segment will be defined over $[u_j, u_{j+1})$ as

$$\gamma_j(t) = \sum_{i=0}^{2} R_{j,i} \theta_{i,2}\left(\frac{t-u_j}{u_{j+1}-u_j} \right),$$



where $R_{j,1} = P_{j+1}$ and for C^0 continuity, $R_{j,2} = R_{j+1,0}$.

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$$\gamma_j(t) = \sum_{i=0}^{2} R_{j,i}\theta_{i,2}\left(\frac{t-u_j}{u_{j+1}-u_j}\right),$$

Constructive Piecewise Curves (Cont.)

C^1 continuity constraint at $t = u_{j+1}$ yields,

$$\gamma'_j(u_{j+1}) = \gamma'_{j+1}(u_{j+1})$$

or,

$$\frac{2}{u_{j+1}-u_j}\left(R_{j,2}-R_{j,1}\right) = \frac{2}{u_{j+2}-u_{j+1}}\left(R_{j+1,1}-R_{j+1,0}\right).$$

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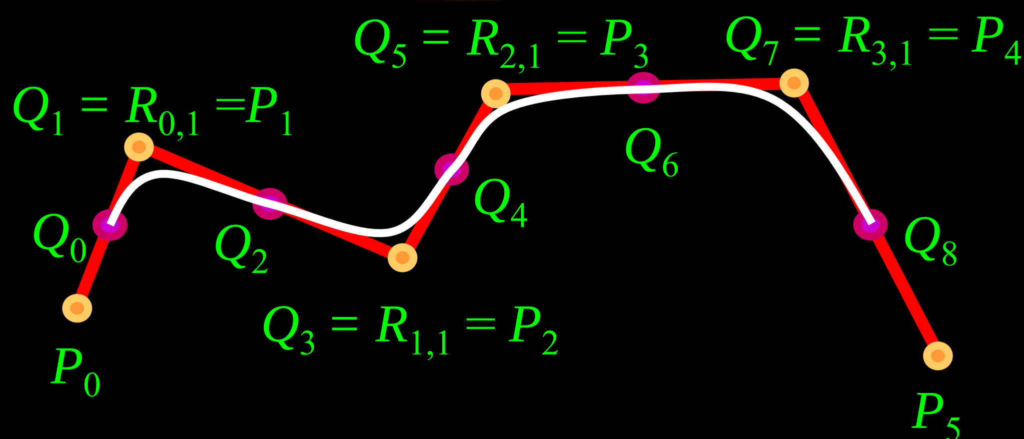
$$\frac{2}{u_{j+1}-u_j}\left(R_{j,2}-R_{j,1}\right)=\frac{2}{u_{j+2}-u_{j+1}}\left(R_{j+1,1}-R_{j+1,0}\right)$$

Constructive Piecewise Curves (Cont.)

Let $Q_{2(j+1)} = R_{j,2} = R_{j+1,0}$ and

$Q_{2j+1} = R_{j,1} = P_{j+1}$,

for $j = 0,\ldots,n\text{-}3$:



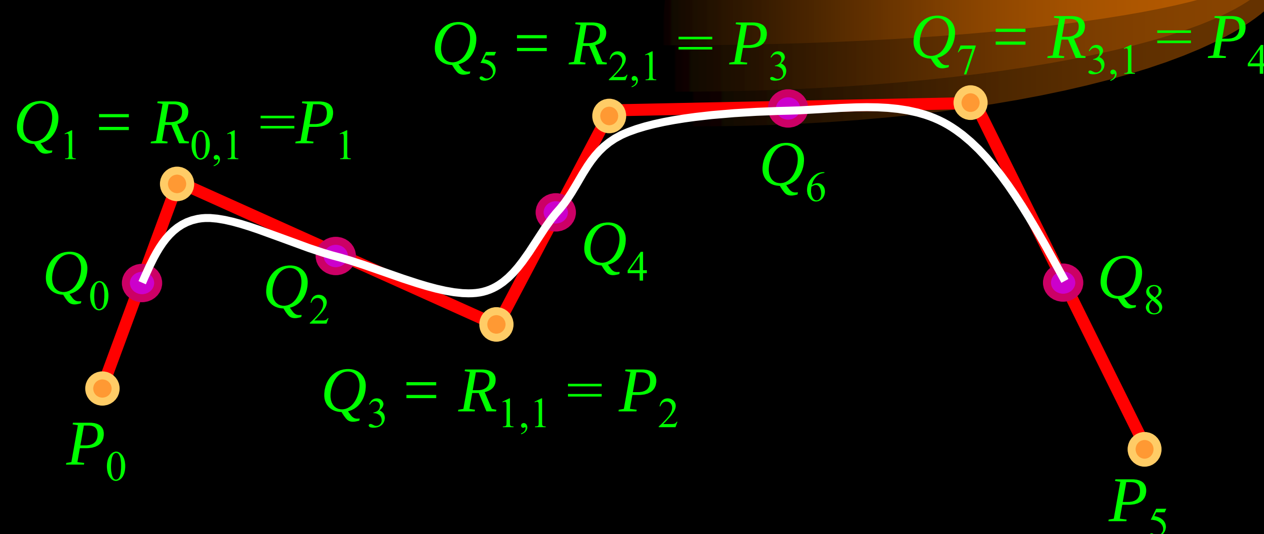
Then, the last equation may be rewritten as

$$\frac{2}{u_{j+1}-u_j}\left(Q_{2(j+1)}-P_{j+1}\right)=\frac{2}{u_{j+2}-u_{j+1}}\left(P_{j+2}-Q_{2(j+1)}\right)$$

$$\frac{2}{u_{j+1}-u_j}\left(Q_{2(j+1)}-P_{j+1}\right)=\frac{2}{u_{j+2}-u_{j+1}}\left(P_{j+2}-Q_{2(j+1)}\right)$$

Constructive Piecewise Curves (Cont.)

Solving for $Q_{2(j+1)}$,



$$Q_{2(j+1)}=\frac{u_{j+2}-u_{j+1}}{u_{j+2}-u_j}P_{j+1}+\frac{u_{j+1}-u_j}{u_{j+2}-u_j}P_{j+2}.$$

Assuming $u_i = i$, yields $Q_{2(j+1)}=\frac{1}{2}P_{j+1}+\frac{1}{2}P_{j+2}.$

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Definition 6.2

A sequence $\boldsymbol{u} = \{ u_i \}_{i=0}^{s}$ of distinct real values is called a breakpoint sequence.

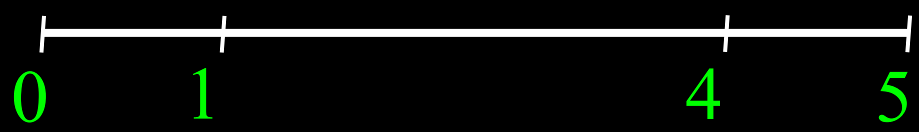
An associated sequence of positive integer values, $\boldsymbol{m} = \{ m_i \}_{i=0}^{s}$, one for each element of $\boldsymbol{u}$, is called the multiplicity vector.

A nondecreasing sequence of real numbers $\boldsymbol{t} = \{ t_i \}$ such that $m_i = card\{ j \mid t_j = u_i \}$ and $t_j \leq t_{j+1}$ is called a knot vector.

Definition 6.2 (Cont.)

Example:

The knot vector $\boldsymbol{t} = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$



has the breakpoint sequence $\boldsymbol{u} = \{ 0, 1, 4, 5 \}$ and the multiplicity vector of $\boldsymbol{m} = \{ 4, 1, 1, 4 \}$.

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Algorithm 6.4 (Recursive B-Spline Alg.)

To define a piecewise polynomial curve of degree k, called $\gamma(t)$, we require the domain to be $[t_k, t_{N-k})$, where $\{ t_i \}_{i=0}^{N}$ is as defined in Definition 6.2.

1. For a given $t_{N-k} > t \geq t_k$, find J such that $t \in [t_J, t_{J+1})$.
2. Define $P_i^{[0]} = P_i$.
3. For $p = 1, \ldots, k$, set

$$P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$$

$$i = J - k + p, \ldots, J.$$

4. Then, $\gamma(t) = P_J^{[k]}$.

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$$P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$$

$$i = J, \ldots, J - k + p.$$

Example 6.5

t_0

Let $k = 3$ and $\boldsymbol{t} = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$. Given $t = 2$, $J = 4$ since $t_4 = 1 \le 2 < t_5 = 4$. We now have,

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Question: What is the domain of the curve?

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$t = \{\ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5\ \}$

$t_0 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Example 6.5 (Cont.)

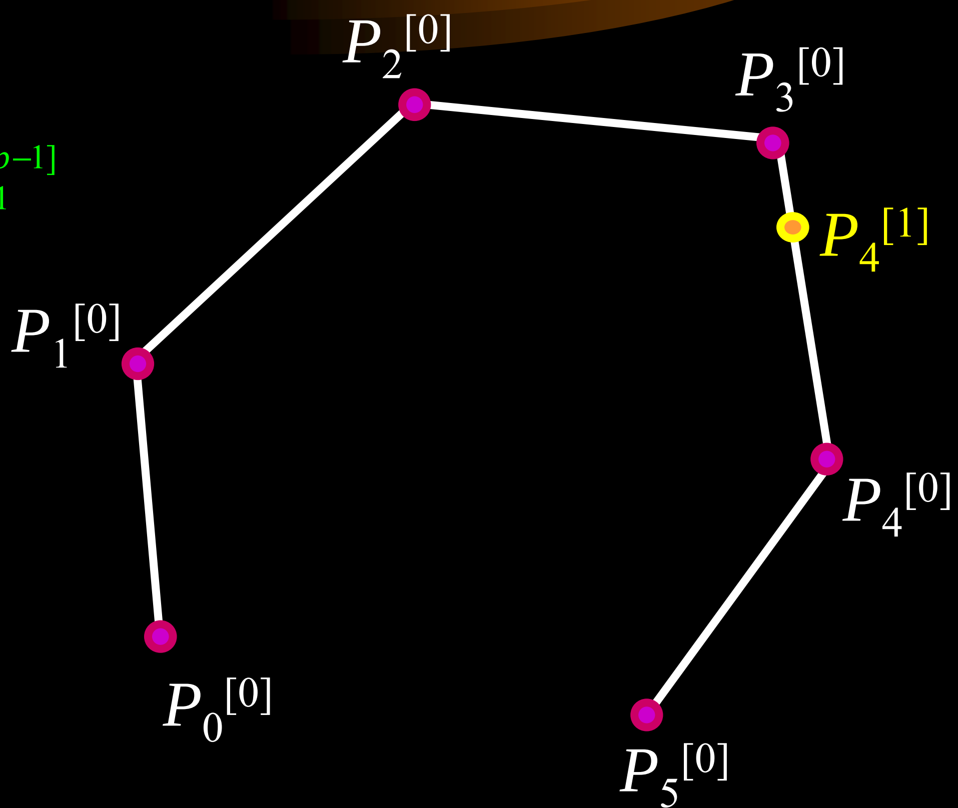
For $k = 3,\ p = 1,\ i = 4,$

$$P_4^{[1]} = \frac{t - t_i}{t_{i+4-1} - t_i} P_i^{[p-1]} + \frac{t_{i+4-1} - t}{t_{i+4-1} - t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t - t_4}{t_{8-1} - t_4} P_4^{[1-1]} + \frac{t_{8-1} - t}{t_{8-1} - t_4} P_3^{[1-1]}$$

$$= \frac{2-1}{5-1} P_4^{[0]} + \frac{5-2}{5-1} P_3^{[0]}$$

$$= \frac{1}{4} P_4^{[0]} + \frac{3}{4} P_3^{[0]}$$



$t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$

$t_0 \quad t_3 \; t_4 \; t_5 \; t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Example 6.5 (Cont.)

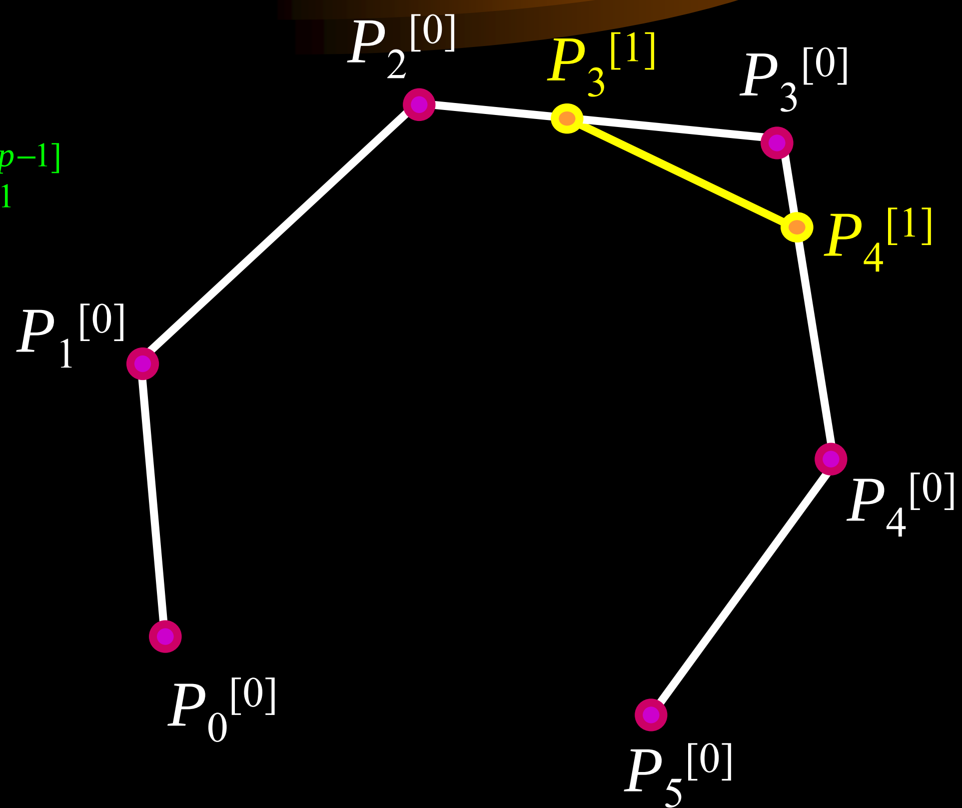
For $k = 3,\ p = 1,\ i = 3,$

$$P_3^{[1]} = \frac{t - t_i}{t_{i+4-1} - t_i} P_i^{[p-1]} + \frac{t_{i+4-1} - t}{t_{i+4-1} - t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t - t_3}{t_{7-1} - t_3} P_3^{[1-1]} + \frac{t_{7-1} - t}{t_{7-1} - t_3} P_2^{[1-1]}$$

$$= \frac{2 - 0}{5 - 0} P_3^{[0]} + \frac{5 - 2}{5 - 0} P_2^{[0]}$$

$$= \frac{2}{5} P_3^{[0]} + \frac{3}{5} P_2^{[0]}$$



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$t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$

$t_0 \quad t_3 \; t_4 \; t_5 \; t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4,3,\ldots,1+p.$$

Example 6.5 (Cont.)

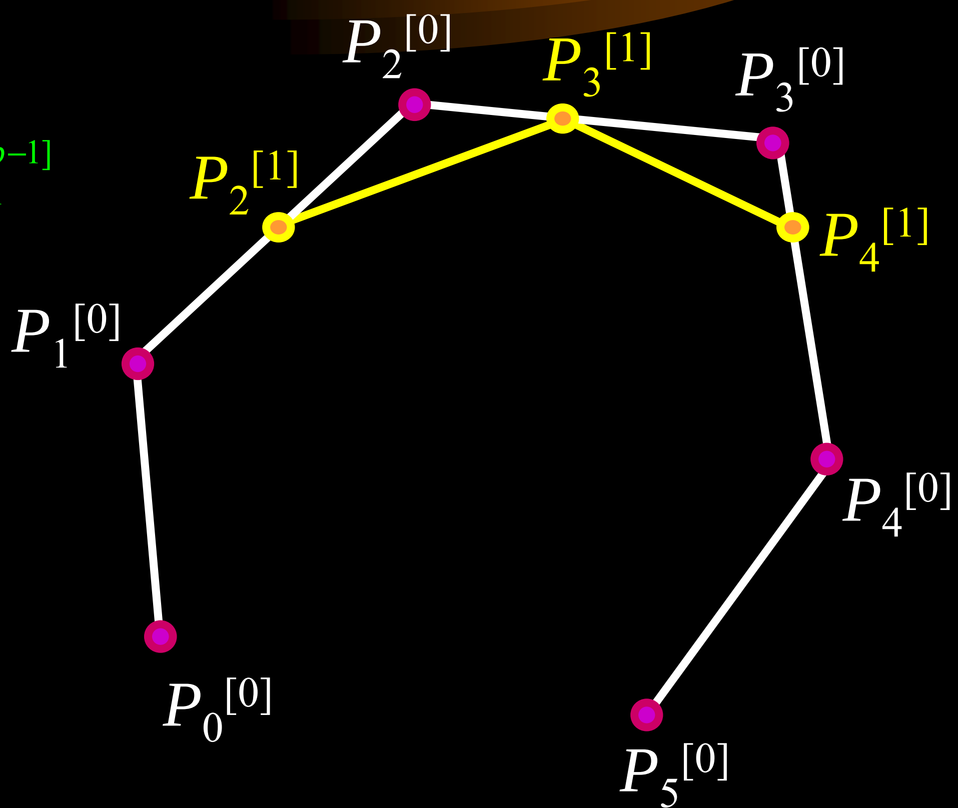
For $k = 3,\ p = 1,\ i = 2,$

$$P_2^{[1]} = \frac{t - t_i}{t_{i+4-1} - t_i} P_i^{[p-1]} + \frac{t_{i+4-1} - t}{t_{i+4-1} - t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t - t_2}{t_{6-1} - t_2} P_2^{[1-1]} + \frac{t_{6-1} - t}{t_{6-1} - t_2} P_1^{[1-1]}$$

$$= \frac{2-0}{4-0} P_2^{[0]} + \frac{4-2}{4-0} P_1^{[0]}$$

$$= \frac{1}{2} P_2^{[0]} + \frac{1}{2} P_1^{[0]}$$



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$t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$

$t_0 \quad t_3 \; t_4 \; t_5 \; t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Example 6.5 (Cont.)

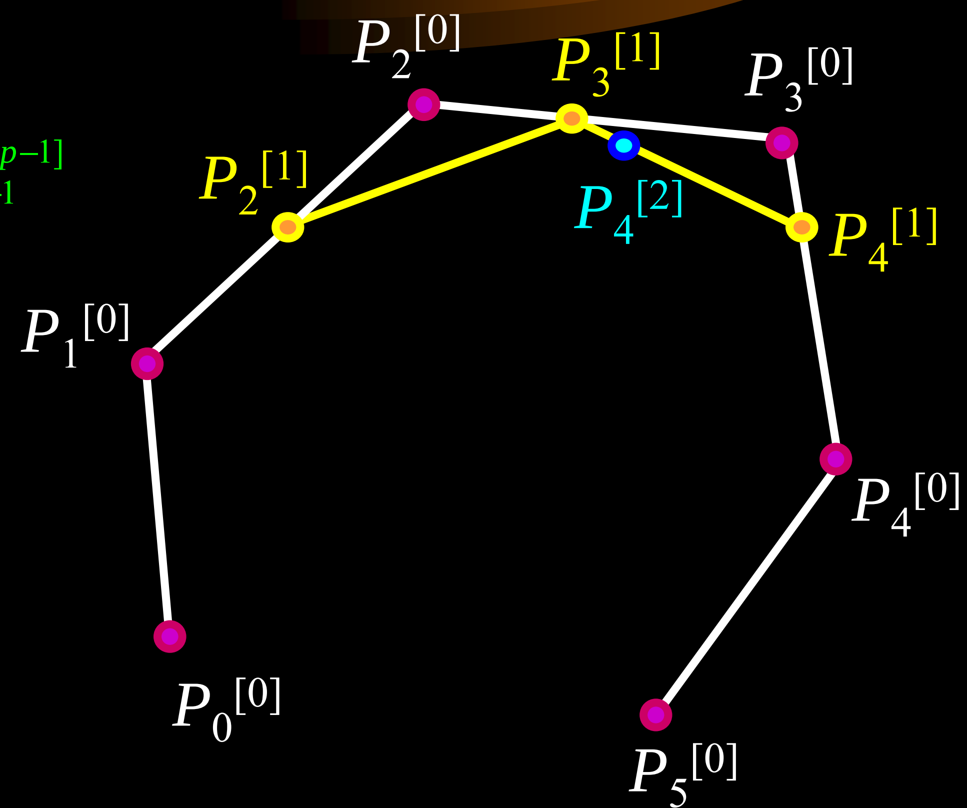
For $k = 3,\ p = 2,\ i = 4,$

$$P_4^{[2]} = \frac{t - t_i}{t_{i+4-2} - t_i} P_i^{[p-1]} + \frac{t_{i+4-2} - t}{t_{i+4-2} - t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t - t_4}{t_{8-2} - t_4} P_4^{[2-1]} + \frac{t_{8-2} - t}{t_{8-2} - t_4} P_3^{[2-1]}$$

$$= \frac{2 - 1}{5 - 1} P_4^{[1]} + \frac{5 - 2}{5 - 1} P_3^{[1]}$$

$$= \frac{1}{4} P_4^{[1]} + \frac{3}{4} P_3^{[1]}$$



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$\boldsymbol{t} = \{\, 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \,\}$

$t_0 \quad t_3 \; t_4 \; t_5 \; t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Example 6.5 (Cont.)

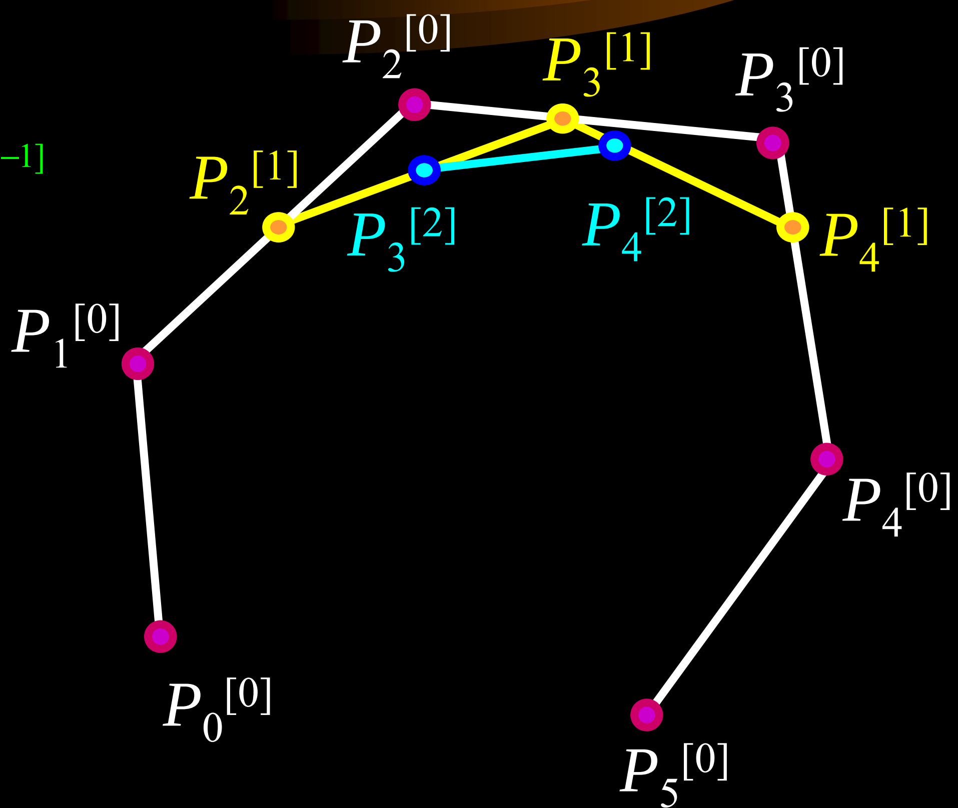
For $k = 3,\ p = 2,\ i = 3,$

$$P_3^{[2]} = \frac{t - t_i}{t_{i+4-2} - t_i} P_i^{[p-1]} + \frac{t_{i+4-2} - t}{t_{i+4-2} - t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t - t_3}{t_{7-2} - t_3} P_3^{[2-1]} + \frac{t_{7-2} - t}{t_{7-2} - t_3} P_2^{[2-1]}$$

$$= \frac{2 - 0}{4 - 0} P_3^{[1]} + \frac{4 - 2}{4 - 0} P_2^{[1]}$$

$$= \frac{1}{2} P_3^{[1]} + \frac{1}{2} P_2^{[1]}$$



$\boldsymbol{t} = \{\ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5\ \}$

$t_0 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_9$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$$i = 4, 3, \ldots, 1 + p.$$

Example 6.5 (Cont.)

For $k = 3,\ p = 3,\ i = 4,$

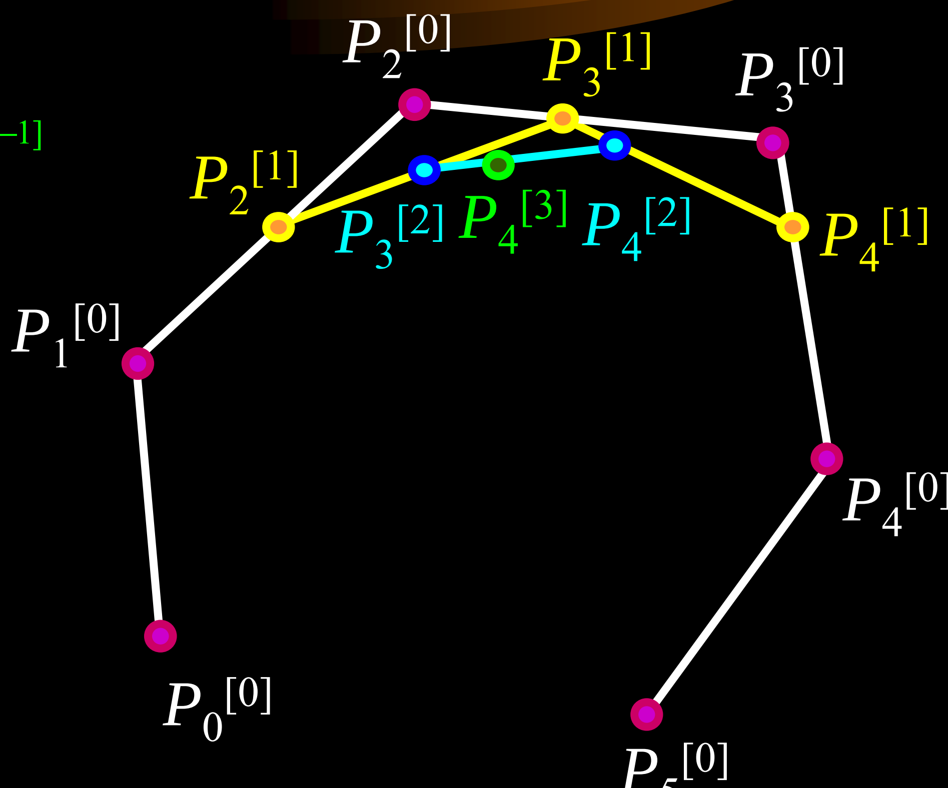
$$P_4^{[3]} = \frac{t - t_i}{t_{i+4-3} - t_i} P_4^{[p-1]} + \frac{t_{i+4-3} - t}{t_{i+4-3} - t_i} P_3^{[p-1]}$$

$$= \frac{t - t_4}{t_{8-3} - t_4} P_4^{[3-1]} + \frac{t_{8-3} - t}{t_{8-3} - t_4} P_3^{[3-1]}$$

$$= \frac{2-1}{4-1} P_4^{[2]} + \frac{4-2}{4-1} P_3^{[2]}$$

$$= \frac{1}{3} P_4^{[2]} + \frac{2}{3} P_3^{[2]}$$

and $\gamma(t) = P_4^{[3]}$.



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Example 6.6

Let $\boldsymbol{t} = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$ and $k = 2$.
The domain of the curve equals $[0, 4)$.

For $t = [0, 1)$, $J = 2$ (why?) and we now have,

1. $P_i^{[0]} = P_i$, for $i = 0, 1, 2$.

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$t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$$

$$i = J, J-1, \ldots, J-k+p.$$

Example 6.6 (Cont.)

Continuing with $p = 1$, $k = 2$, $t = [0, 1)$, $J = 2$, $i = 2,\ldots,1$,

2.

$$P_2^{[1]} = \frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_{2-1}^{[0]},$$

$$P_1^{[1]} = \frac{t - t_1}{t_{1+2} - t_1} P_1^{[0]} + \frac{t_{1+2} - t}{t_{1+2} - t_1} P_{1-1}^{[0]}.$$

For $p = 2$, $k = 2$, $t = [0, 1)$, $J = 2$, $i = 2,\ldots,2$,

3.

$$P_2^{[2]} = \frac{t - t_2}{t_{2+1} - t_2} P_2^{[1]} + \frac{t_{2+1} - t}{t_{2+1} - t_2} P_{2-1}^{[1]}.$$

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$t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$

Example 6.6 (Cont.)

Thus, for $t = [0, 1)$, the curve $\gamma(t)$ is quadratic,

$$\gamma(t) = \frac{t - t_2}{t_{2+1} - t_2}\left(\frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_{2-1}^{[0]}\right) + \frac{t_{2+1} - t}{t_{2+1} - t_2}\left(\frac{t - t_1}{t_{1+2} - t_1} P_1^{[0]} + \frac{t_{1+2} - t}{t_{1+2} - t_1} P_{1-1}^{[0]}\right).$$

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$t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$$

$$i = J, J-1, \ldots, J-k+p.$$

Example 6.6 (Cont.)

Similarly, for $t = [1, 3)$, $J = 3$ (why?) and we now have,

1. $P_i^{[0]} = P_i$, for $i = 1, 2, 3$ and recall that $k = 2$.
2. Continuing with $p = 1$, i=3,...,2:

$$P_3^{[1]} = \frac{t - t_3}{t_{3+2} - t_3} P_3^{[0]} + \frac{t_{3+2} - t}{t_{3+2} - t_3} P_{3-1}^{[0]},$$

$$P_2^{[1]} = \frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_{2-1}^{[0]}.$$

3. And $p = 2$, i=3,...,3:

$$P_3^{[2]} = \frac{t - t_3}{t_{3+1} - t_3} P_3^{[1]} + \frac{t_{3+1} - t}{t_{3+1} - t_3} P_{3-1}^{[1]}.$$

$t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$

Example 6.6 (Cont.)

Thus, for $t = [1, 3)$, the curve $\gamma(t)$ is quadratic as well,

$$\gamma(t) = \frac{t - t_3}{t_{3+1} - t_3}\left(\frac{t - t_3}{t_{3+2} - t_3} P_3^{[0]} + \frac{t_{3+2} - t}{t_{3+2} - t_3} P_{3-1}^{[0]} \right) + \frac{t_{3+1} - t}{t_{3+1} - t_3}\left(\frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_{2-1}^{[0]} \right).$$

A similar result could be obtained for $t = [3, 4)$.

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The B-spline Blending Functions

We seek blending functions that extend the properties of Bezier curves to piecewise polynomials.

Following the recursive Bezier Basis functions' scheme we seek functions of degree k that satisfy

$$\gamma(t) = \sum P_i B_{i,k}(t)$$
$$= \sum P_i^{[j]} B_{i,k-j}(t) \qquad j = 0, \ldots, k.$$

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The B-spline Blending Functions (Cont.)

For a given t, there are only $k+1$ values of $P_i^{[0]}$, $i = J - k, \ldots, J$ that contribute, so,

$$\gamma(t) = \sum P_i^{[j]} B_{i,k-j}(t) \qquad j = 0,\ldots,k$$

$$= \sum \left(\frac{t - t_i}{t_{i+k-j+1} - t_i} P_i^{[j-1]} + \frac{t_{i+k-j+1} - t}{t_{i+k-j+1} - t_i} P_{i-1}^{[j-1]} \right) B_{i,k-j}(t)$$

$$= \sum P_i^{[j-1]} \left(\frac{t - t_i}{t_{i+k-j+1} - t_i} B_{i,k-j}(t) + \frac{t_{i+1+k-j+1} - t}{t_{i+1+k-j+1} - t_{i+1}} B_{i+1,k-j}(t) \right)$$

$$= \sum P_i^{[j-1]} B_{i,k-j+1}(t).$$

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The B-spline Blending Functions (Cont.)

Hence, and as we seek $\gamma(t) = \Sigma P_i^{[j-1]} B_{i,k-(j-1)}(t)$ with a recursive form, we end up with,

$$B_{i,k-(j-1)}(t) = \left(\frac{t - t_i}{t_{i+k-j+1} - t_i} B_{i,k-j}(t) + \frac{t_{i+1+k-j+1} - t}{t_{i+1+k-j+1} - t_{i+1}} B_{i+1,k-j}(t) \right)$$

or letting $r = k - j$,

$$B_{i,r+1}(t) = \left(\frac{t - t_i}{t_{i+r+1} - t_i} B_{i,r}(t) + \frac{t_{i+r+2} - t}{t_{i+r+2} - t_{i+1}} B_{i+1,r}(t) \right).$$

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Definition 6.7

Let $t_0 \le t_1 \le \ldots \le t_N$ be a sequence of real numbers.

For $k = 0,\ldots,N$-1, and $i = 0,\ldots,N-(k+1)$, define the i'th (normalized) B-spline $B_{i,k}$ of degree k as

$$B_{i,0}(t) = \begin{cases} 1 & for \quad t_i \le t < t_{i+1}, \\ 0 & otherwise, \end{cases}$$

and for $k > 0$,

$$B_{i,k}(t) = \begin{cases} \dfrac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \dfrac{t_{i+1+k} - t}{t_{i+1+k} - t_{i+1}} B_{i+1,k-1}(t), & t_i \le t < t_{i+k+1}, \\ 0 & otherwise. \end{cases}$$

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$$B_{i,0}(t) = \begin{cases} 1 & for \quad t_i \le t < t_{i+1}, \\ 0 & otherwise, \end{cases}$$

$$B_{i,k}(t) = \begin{cases} \dfrac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \dfrac{t_{i+1+k} - t}{t_{i+1+k} - t_{i+1}} B_{i+1,k-1}(t), & t_i \le t < t_{i+k+1}, \\ 0 & otherwise. \end{cases}$$

Consider the case of $k = 0$ and assume $t_i = t_{i+1}$.

Question: What is the shape of $B_{i,0}(t)$?

Consider $B_{i,k}(t)$ and assume $t_i = \ldots = t_{i+k+1}$.

Question: What is the shape of $B_{i,k}(t)$?

Consider $B_{i,k}(t)$ and assume $t_i = \ldots = t_{i+k} < t_{i+k+1}$.

Question: What is the shape of $B_{i,k}(t)$?

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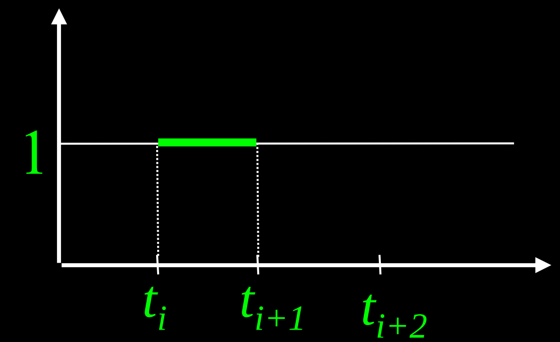
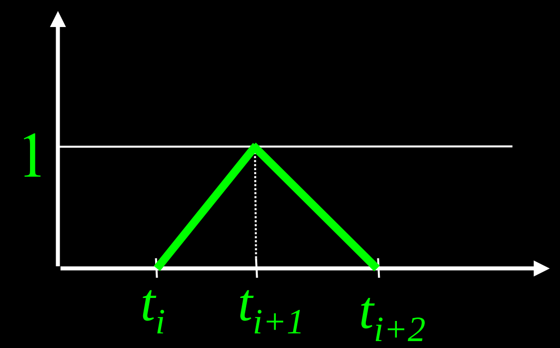
Example 6.8

Question: What is the shape of $B_{i,1}(t)$?

$$B_{i,1}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,0}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,0}(t)$$

$$= \begin{cases} \dfrac{t - t_i}{t_{i+1} - t_i} & \text{for} \quad t_i \le t < t_{i+1} \\ \dfrac{t_{i+2} - t}{t_{i+2} - t_{i+1}} & \text{for} \quad t_{i+1} \le t < t_{i+2}. \end{cases}$$





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Lemma 6.9

If $t \geq t_{i+1+k}$ or $t < t_i$, then $B_{i,k}(t) = 0$; i.e., $B_{i,k}(t)$ can be nonzero only on the interval $[t_i , t_{i+k+1})$.

Proof

By induction.

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Lemma 6.10

$$B_{i,k}(t) > 0 \quad \text{for } t \in (t_i, t_{i+k+1}).$$

Further, $B_{i,k}(t_{i+k+1}) = 0.$

Proof

By definition $B_{i,0}(t) > 0$ for $t \in (t_i, t_{i+1})$. Further, $B_{i,0}(t_{i+1}) = 0$ and hence Lemma 6.10 holds for $k = 0$.

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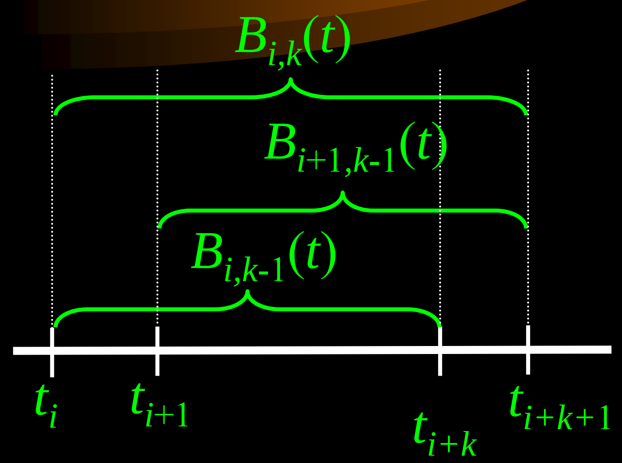
$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+1+k} - t}{t_{i+1+k} - t_{i+1}} B_{i+1,k-1}(t).$$

Lemma 6.10 (Cont.)



Assume Lemma 6.10 holds for degree k-1. Consider degree k.

Question: What is the support of $B_{i,k-1}(t)$? of $B_{i+1,k-1}(t)$? of $B_{i,k}(t)$?

$$B_{i,k}(t_{i+k+1}) = \frac{t_{i+k+1} - t_i}{t_{i+k} - t_i} B_{i,k-1}(t_{i+k+1}) + \frac{t_{i+k+1} - t_{i+k+1}}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t_{i+k+1})$$

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Definition 6.11

A function which is non-zero only over a finite interval of the real line is called a local function.

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Corollary 6.12

If $t \in (t_j, t_{j+1})$, then $B_{i,k}(t) > 0$ for

$$i \in \{ j\text{-}k, j\text{-}k+1, \ldots, j \}.$$

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Theorem 6.13

If $\gamma(t)$ is the curve algorithmically defined in Algorithm 6.4 and $\alpha(t) = \Sigma P_i B_{i,k}(t)$, where the $B_{i,k}(t)$ are defined in Definition 6.7, then $\gamma(t) = \alpha(t)$.

Proof

$$\alpha(t) = \sum_{m=0}^{n} P_m B_{m,k}(t) = \sum_{m=J-k}^{J} P_m B_{m,k}(t), \qquad t \in [t_J, t_{J+1}),$$

since only non zero B-splines contribute.

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$$\alpha(t) = \sum_{m=0}^{n} P_m B_{m,k}(t) = \sum_{m=J-k}^{J} P_m B_{m,k}(t), \qquad t \in [t_J, t_{J+1}),$$

Theorem 6.13 (Cont.)

Which, when using the recursive definition equals,

$$= \sum_{m=J-k}^{J} P_m \left[\frac{t_{m+k+1} - t}{t_{m+k+1} - t_{m+1}} B_{m+1,k-1}(t) + \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t) \right]$$

$$= \sum_{m=J-k}^{J} P_m \frac{t_{m+k+1} - t}{t_{m+k+1} - t_{m+1}} B_{m+1,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t)$$

$$= \sum_{m=J+1-k}^{J+1} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t).$$

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$t \in [t_J, t_{J+1})$

Theorem 6.13 (Cont.)

Inspecting the last line,

$$\sum_{m=J+1-k}^{J+1} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t),$$

the last term of the first summation and the first term of the last summation equal zero (why?), or,

$$\alpha(t) = \sum_{m=J+1-k}^{J} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J+1-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t)$$

$$= \sum_{m=J+1-k}^{J} \left[\frac{t_{m+k} - t}{t_{m+k} - t_m} P_{m-1} + \frac{t - t_m}{t_{m+k} - t_m} P_m \right] B_{m,k-1}(t).$$

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$$\alpha(t) = \sum_{m=J+1-k}^{J} \left[\frac{t_{m+k} - t}{t_{m+k} - t_m} P_{m-1} + \frac{t - t_m}{t_{m+k} - t_m} P_m \right] B_{m,k-1}(t).$$

Theorem 6.13 (Cont.)

Therefore,

$$\alpha(t) = \sum_{m=J+1-k}^{J} P_m^{[1]} B_{m,k-1}(t)$$

Iterating this process k-1 times yields.

$$\alpha(t) = \sum_{m=J}^{J} P_m^{[k]} B_{m,0}(t) = P_J^{[k]} = \gamma(t).$$

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Corollary 6.14

For $t_i \leq t < t_{i+1}$, the B-spline curve is a convex combination of $P_{i-k}, \ldots, P_i$. This is called the convex hull property.

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Corollary 6.16

For a knot vector $\boldsymbol{t} = \{ t_i \}_{i=0}^{N}$, and for $t \in [t_k, t_{N-k})$,

$$\sum_{i=0}^{N-(k+1)} B_{i,k}(t) = 1, \qquad \forall\, k \geq 0.$$

Further, for $t < t_k$ or $t \geq t_{N-k}$, $\sum_{i=0}^{N-(k+1)} B_{i,k}(t) < 1.$

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Corollary 6.16 (Cont.)

Proof

Again, by induction on the degree k. Let,

$j \in \{ k,\ldots,N-(k+1) \}$.

Question: What do we have for $k = 0$?

Now assume $\Sigma B_{i,k-1}(t) = 1$. For $t \in [t_j, t_{j+1})$, and $k > 0$,

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Corollary 6.16 (Cont.)

$$\sum_i B_{i,k}(t) = \sum_{i=j-k}^{j} B_{i,k}(t)$$

$$= \sum_{i=j-k}^{j} \left[\frac{t-t_i}{t_{i+k}-t_i} B_{i,k-1}(t) + \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}} B_{i+1,k-1}(t) \right]$$

$$= \sum_{i=j-k}^{j} \frac{t-t_i}{t_{i+k}-t_i} B_{i,k-1}(t) + \sum_{i=j-k}^{j} \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}} B_{i+1,k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} \frac{t-t_i}{t_{i+k}-t_i} B_{i,k-1}(t) + \sum_{i=j-k}^{j-1} \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}} B_{i+1,k-1}(t),$$

as this time the first term of the first summation and the last term of the last summation equal zero.

$$\sum_i B_{i,k}(t) = \sum_{i=j-k+1}^{j} \frac{t-t_i}{t_{i+k}-t_i} B_{i,k-1}(t) + \sum_{i=j-k}^{j-1} \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}} B_{i+1,k-1}(t),$$

Corollary 6.16 (Cont.)

$$= \sum_{i=j-k+1}^{j} \frac{t-t_i}{t_{i+k}-t_i} B_{i,k-1}(t) + \sum_{i=j+1-k}^{j} \frac{t_{i+k}-t}{t_{i+k}-t_i} B_{i,k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} \left[\frac{t-t_i}{t_{i+k}-t_i} + \frac{t_{i+k}-t}{t_{i+k}-t_i} \right] B_{i,k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} B_{i,k-1}(t)$$

$$= \sum_{i=0}^{N-k} B_{i,k-1}(t)$$

$= 1,$ by the induction hypothesis.

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Corollary 6.16 (Cont.)

Question: What about for $t < t_k$?

By selecting values $t_{-k} \leq \ldots \leq t_{-1} < t_0$ and considering the augmented knot vector $\{t_{-k}, \ldots, t_{-1}, t_0, t_1 \ldots, t_N\}$, we have for $t \in [t_0, t_{k-1})$:

$$1 = \sum_{i=-k}^{N-(k+1)} B_{i,k}(t) = \sum_{i=-k}^{-1} B_{i,k}(t) + \sum_{i=0}^{N-(k+1)} B_{i,k}(t).$$

Now since the first sum is positive, the second must be less than one.

Theorem 6.17 (Derivatives)

For $B_{i,k}(t)$ a B-spline basis function of degree k defined by the knot vector $\boldsymbol{t}$, when the derivative exists (i.e., between knots and sometimes at knots), it is defined by

$$B'_{i,k}(t) = k\left[\frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}\right].$$

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Theorem 6.17 (Cont.)

Proof

Once again, the proof is by induction:

$$B_{i,0}(t) = \begin{cases} 1 & \text{for} \quad t_i \leq t < t_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

and hence $B_{i,0}^{'}(t) = 0$ for all t but $t = t_i$ and $t = t_{i+1}$, where is it discontinuous.

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$$B'_{i,k}(t) = k\left[\frac{B_{i,k-1}(t)}{t_{i+k}-t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1}-t_{i+1}}\right].$$

Theorem 6.17 (Cont.)

Then, consider $k = 1$ (assuming $t_i < t_{i+2}$),

$$B_{i,1}(t) = \frac{t-t_i}{t_{i+1}-t_i}B_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}B_{i+1,0}(t),$$

so for $t \notin \{t_i, t_{i+1}, t_{i+2}\}$,

$$B'_{i,1}(t) = \frac{1}{t_{i+1}-t_i}B_{i,0}(t) - \frac{1}{t_{i+2}-t_{i+1}}B_{i+1,0}(t)$$

$$+\frac{t-t_i}{t_{i+1}-t_i}B'_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}B'_{i+1,0}(t)$$

$$= \frac{1}{t_{i+1}-t_i}B_{i,0}(t) - \frac{1}{t_{i+2}-t_{i+1}}B_{i+1,0}(t),$$ following Theorem 6.17.

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Theorem 6.17 (Cont.)

Now assume the theorem holds for degree k-1, for $t \notin \{t_j, \ldots, t_{j+k-1}\}$, and prove it for degree k.

Differentiating $B_{i,k}(t)$:

$$B'_{i,k}(t) = \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} + \frac{t - t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t).$$

We now examine the last two terms only,

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$$B'_{i,k}(t) = k\left[\frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}\right].$$

Theorem 6.17 (Cont.)

Examining the last two terms only,

$$\frac{t - t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t)$$

$$= \frac{t - t_i}{t_{i+k} - t_i}(k-1)\left[\frac{1}{t_{i+k-1} - t_i} B_{i,k-2}(t) - \frac{1}{t_{i+k} - t_{i+1}} B_{i+1,k-2}(t)\right]$$

$$+ \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}}(k-1)\left[\frac{1}{t_{i+k} - t_{i+1}} B_{i+1,k-2}(t) - \frac{1}{t_{i+k+1} - t_{i+2}} B_{i+2,k-2}(t)\right],$$

by the induction hypothesis for this theorem.

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Theorem 6.17 (Cont.)

$$\frac{t-t_i}{t_{i+k}-t_i}B'_{i,k-1}(t)+\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}B'_{i+1,k-1}(t)$$
$$=\frac{t-t_i}{t_{i+k}-t_i}(k-1)\left[\frac{1}{t_{i+k-1}-t_i}B_{i,k-2}(t)-\frac{1}{t_{i+k}-t_{i+1}}B_{i+1,k-2}(t)\right]$$
$$+\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}(k-1)\left[\frac{1}{t_{i+k}-t_{i+1}}B_{i+1,k-2}(t)-\frac{1}{t_{i+k+1}-t_{i+2}}B_{i+2,k-2}(t)\right],$$

$$=(k-1)\left[\frac{1}{t_{i+k}-t_i}\left(\frac{t-t_i}{t_{i+k-1}-t_i}B_{i,k-2}(t)+\frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}B_{i+1,k-2}(t)\right)\right]$$
$$-(k-1)\frac{1}{t_{i+k}-t_{i+1}}\left[\frac{t_{i+k}-t}{t_{i+k}-t_i}+\frac{t-t_i}{t_{i+k}-t_i}-\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}\right]B_{i+1,k-2}(t)$$
$$-(k-1)\frac{1}{t_{i+k+1}-t_{i+1}}\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+2}}B_{i+2,k-2}(t)$$

$$1-\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}=\frac{t-t_{i+1}}{t_{i+k+1}-t_{i+1}}$$

$$=(k-1)\left[\frac{1}{t_{i+k}-t_i}B_{i,k-1}(t)\right]$$
$$-(k-1)\frac{1}{t_{i+k+1}-t_{i+1}}\left[\frac{t-t_{i+1}}{t_{i+k}-t_{i+1}}B_{i+1,k-2}(t)+\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+2}}B_{i+2,k-2}(t)\right]$$
$$=(k-1)\left[\frac{1}{t_{i+k}-t_i}B_{i,k-1}(t)-\frac{1}{t_{i+k+1}-t_{i+1}}B_{i+1,k-1}(t)\right]$$

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Theorem 6.17 (Cont.)

Going back to the original differentiation of $B_{i,k}(t)$:

$$B'_{i,k}(t) = \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}$$

$$+ \frac{t - t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t)$$

$$= \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} + (k-1)\left[\frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}\right]$$

$$= k\left[\frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}\right],$$

completing the proof.

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Theorem 6.18

For B-spline curve $\gamma(t) = \sum_{i=0}^{n} P_i B_{i,k}(t)$, the jth derivative is given by

$$\frac{d^j}{dt^j}\left(\gamma(t)\right) = \sum_{i=j}^{n} Q_{j,i} B_{i,k-j}(t)$$

where

$$Q_{j,i} = \begin{cases} P_i, & \text{for } j = 0, \\ (k-j+1)\dfrac{Q_{j-1,i} - Q_{j-1,i-1}}{t_{i+k-j+1} - t_i}, & \text{for } j > 0. \end{cases}$$

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Theorem 6.18 (Cont.)

Proof

For $j=0$, the theorem clearly holds.

Assume the theorem holds for all lower degree derivatives and prove the theorem for the jth derivative.

Then,

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Theorem 6.18 (Cont.)

$$\frac{d^j}{dt^j}\big(\gamma(t)\big) = \frac{d}{dt}\left(\frac{d^{j-1}}{dt^{j-1}}\big(\gamma(t)\big)\right)$$

$$= \frac{d}{dt}\left(\sum Q_{j-1,i} B_{i,k-j+2-1}(t)\right)$$

$$= \sum Q_{j-1,i} \frac{d}{dt} B_{i,k-j+2-1}(t)$$

$$= \sum Q_{j-1,i} (k-j+1)\left(\frac{B_{i,k-j}}{t_{i+k-j+1}-t_i} - \frac{B_{i+1,k-j}}{t_{i+k-j+2}-t_{i+1}}\right),$$

which follows from Theorem 6.17. This proof is completed by expanding the summation and regrouping.

Corollary 6.21 (Integrals)

$$\int_{-\infty}^{t} B_{i,k}(u)du = \begin{cases} 0, & t < t_i \\ \sum_{j=i}^{n} \frac{t_{i+1+k} - t_i}{k+1} B_{j,k+1}(t), & t_i \le t < t_{n+1} \\ \frac{t_{i+1+k} - t_i}{k+1}, & t_{n+1} \le t \end{cases}$$

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Corollary 6.21 (Integrals, Cont.)

Proof

By integrating the derivative equation of

$$B_{i,k}'(u) = k\left(\frac{B_{i,k-1}(u)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(u)}{t_{i+k+1} - t_{i+1}}\right)$$

And rearranging:

$$\frac{1}{t_{i+k} - t_i}\int B_{i,k-1}(u) = \frac{B_{i,k}(u)}{k} + \frac{1}{t_{i+k+1} - t_{i+1}}\int B_{i+1,k-1}(u)$$

$$\frac{1}{t_{i+k}-t_i}\int B_{i,k-1}(u) = \frac{B_{i,k}(u)}{k} + \frac{1}{t_{i+k+1}-t_{i+1}}\int B_{i+1,k-1}(u)$$

Corollary 6.21 (Integrals – Cont.)

Or $$\int B_{i,k}(u) = \frac{(t_{i+k+1}-t_i)B_{i,k+1}(u)}{k+1} + \frac{t_{i+k+1}-t_i}{t_{i+k+2}-t_{i+1}}\int B_{i+1,k}(u).$$

For the last non-zero interval ($i = n$, $B_{n+1,k}(t) = 0$), we have,

$$\int B_{n,k}(u) = \frac{(t_{i+k+1}-t_i)B_{n,k+1}(u)}{k+1}.$$

Or

$$\int B_{i,k}(u) = \frac{1}{k+1}\sum_{j=i}^{n}(t_{i+k+1}-t_i)B_{j,k+1}(u) = \frac{t_{i+k+1}-t_i}{k+1}\sum_{j=i}^{n}B_{j,k+1}(u).$$

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$$\int B_{i,k}(u) = \frac{t_{i+k+1} - t_i}{k+1} \sum_{j=i}^{n} B_{j,k+1}(u).$$

Corollary 6.22 (Integrals of Curves)

The integral of a curve equals,

$$\int_{-\infty}^{t} \sum_{i=0}^{n} P_i B_{i,k}(u)du = \sum_{j=0}^{n} Q_j B_{j,k+1}(t),$$

where

$$Q_j = \sum_{i=0}^{j} \frac{(t_{i+k+1} - t_i)}{k+1} P_i.$$

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$$\int B_{i,k}(u) = \frac{t_{i+k+1} - t_i}{k+1} \sum_{j=i}^{n} B_{j,k+1}(u).$$

Corollary 6.22 (Integrals of Curves)

Proof (recall $\sum_{i=0}^{n}\sum_{j=i}^{n} \bullet = \sum_{j=0}^{n}\sum_{i=0}^{j} \bullet$):

$$\int_{-\infty}^{t} \sum_{i=0}^{n} P_i B_{i,k}(u)du = \sum_{i=0}^{n} P_i \int B_{i,k}(u)du$$

$$= \sum_{i=0}^{n} P_i \frac{\left(t_{i+k+1} - t_i\right)}{k+1} \sum_{j=i}^{n} B_{j,k+1}(t)$$

$$= \frac{1}{k+1} \sum_{j=0}^{n} \left(\sum_{i=0}^{j} \left(t_{i+k+1} - t_i\right) P_i \right) B_{j,k+1}(t).$$

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Definition 6.23 (Piecewise Polynomials)

If $\boldsymbol{u}$ and $\boldsymbol{m}$ are defined as in Definition 6.2, then

$$PP_{k,u,m} = \{\, f(t) \;:\; f(t)\,|_{(u_i,\, u_{i+1})} \text{ is a polynomial of degree less than or equal to } k \text{ and } f \in C^{(k-m_i)} \text{ at } u_i \}.$$

$PP_{k,u,m}$ is the collection of all piecewise polynomials of degree k with continuity k-m_i at u_i.

In other words, $PP_{k,u,m}$ is a vector space.

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Theorem 6.24

If $\{t_j\}$ is a knot sequence developed from $\{u_j\}$ and $\{m_j\}$, as in Definition 6.2, then $B_{i,k} \in PP_{k,\boldsymbol{u},\boldsymbol{m}}$.

Proof

For $k = 0$, $B_{i,0}(t)$ is continuous at neither t_i nor t_{i+1} and Theorem 6.24 holds, having continuity $k\text{-}m_i=0\text{-}1=\text{-}1$ or continuity $C^{(-1)}$. Note $C^{(-j)}$, $j > 1$ denotes $C^{(-1)}$.

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Theorem 6.24 (Cont.)

If $t_i = \ldots = t_{i+k+1}$, then $B_{i,k}(t) \equiv 0$ and the theorem is true. Otherwise, the proof is by induction on the order. If $t_i < t_{i+k}$ ($B_{i,k-1}(t) > 0$) and $t_{i+1} < t_{i+k+1}$ ($B_{i+1,k-1}(t) > 0$),

$$B'_{i,k}(t) = (k-1)\left[\frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}\right],$$

or a linear combination of $C^{(k-1-m_i)}$ functions so $B_{i,k}(t)$, as the integral, is of class $C^{(k-m_i)}$.

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Corollary 6.25

Given knot vector $\boldsymbol{t}$ that has distinct increasing values $\boldsymbol{u} = \{ u_i \}_{i=0}^{s}$, each repeated with multiplicity $\boldsymbol{m} = \{ m_i \}_{i=0}^{s}$, then $B_{i,k}(t)$ is contained in $C^{(k-m_p)}$ at u_p.

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Theorem 6.26

$B_{i,k}(t)$ is unimodal. That is, for $t \in (t_i, t_{i+k+1})$, there exists one extremal point.

Proof

If $t_{i+1} = t_{i+k}$,

$$B'_{i,k}(t) = \begin{cases} k\dfrac{B_{i,k-1}(t)}{t_{i+k} - t_i}, & t \in (t_i, t_{i+1}) \\ -k\dfrac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}, & t \in (t_{i+1}, t_{i+k+1}). \end{cases}$$

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Theorem 6.26 (Cont.)

Both $B_{i,k-1}(t)$ and $B_{i+1,k-1}(t)$ are not zeros in their open and disjoint domains.

Question: What are the extremals points of $B_{i,k}(t)$?

If, in addition, $t_i = t_{i+1}$, then $B_{i,k-1}(t) \equiv 0$, and $B_{i,k}(t)$ is decreasing. Similarly, if $t_{i+1} = t_{i+k+1}$, $B_{i,k}(t)$ is increasing. Finally, if $t_i < t_{i+1} = t_{i+k} < t_{i+k+1}$, $B_{i,k}(t)$ is increasing for $t < t_{i+1}$, and $B_{i,k}(t)$ is decreasing for $t > t_{i+k}$.

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Theorem 6.26 (Cont.)

Now, if $t_{i+1} < t_{i+k}$, $B_{i,k}'(t)$ is continuous for $t \in (t_i, t_{i+k+1})$, and the extremal points of $B_{i,k}(t)$ can occur at the zeros of $B_{i,k}'(t)$ and nowhere else (we consider an open domain).

Because $B_{i,k}(t_i) = B_{i,k}(t_{i+k+1}) = 0$, and using the mean value's theorem, there exists at least one extremal point for $B_{i,k}'(t)$. Using properties of roots of spline curves, one can show that this extremal point is unique.

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Theorem 6.29

If $\boldsymbol{u} = \{a, b\}$ and $\boldsymbol{m} = \{k+1, k+1\}$, let the knot vector be $\boldsymbol{t} = \{ t_0 = t_1 = \ldots = t_k < t_{k+1} = \ldots = t_{2k+1} \}$, where $t_0 = a$ and $t_{k+1} = b$.

Then, for $t \in [a, b]$, $p = 1,\ldots,k$ and $i = 0,\ldots,p$,

$$B_{i+k-p,p}(t) = \theta_{i,p}\left(\frac{t-a}{b-a}\right),$$

showing that the Bernstein/Bezier blending functions are a special B-spline case.

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Theorem 6.29 (Cont.)

Proof:

If $p = 0$, $B_{i+k,0}(t)$ has the support of $[t_{i+k}, t_{i+k+1}]$. Hence,

$$B_{k,0}(t) = 1 = \theta_{0,0}\left(\frac{t-a}{b-a}\right),$$

Or the case of $p = 0$ is proved ($t_{i+k} < t_{i+k+1}$ for $i=0$ only).

We now only consider the general case of $0 < i < p$ (the other cases are left as an exercise):

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$\boldsymbol{t} = \{ t_0 = t_1 = \ldots = t_k = a < b = t_{k+1} = \ldots = t_{2k+1} \}$

Theorem 6.29 (Cont.)

$t_0 = t_1 = \ldots = t_k = a,$

$t_{k+1} = \ldots = t_{2k+1} = b.$

$i = 1, \ldots, p\text{-}1.$

$$
\begin{aligned}
B_{i+k-p,p}(t) &= \frac{t - t_{i+k-p}}{t_{i+k} - t_{i+k-p}} B_{i+k-p,p-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+k+1-p}} B_{i+k+1-p,p-1}(t) \\
&= \frac{t-a}{b-a} \theta_{i-1,p-1}\left(\frac{t-a}{b-a}\right) + \frac{b-t}{b-a} \theta_{i,p-1}\left(\frac{t-a}{b-a}\right) \\
&= \theta_{i,p}\left(\frac{t-a}{b-a}\right).
\end{aligned}
$$

by the induction hypothesis

Question: What is going on for $i = 0$? $i = p$?

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