Computer Aided Geometric Design

B-spline Curves

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based on a book by Cohen, Riesenfeld, & Elber
Constructive Piecewise Curves

Consider an arbitrary polygon, \( \{ P_i \}_{i=0}^n \).

Let \( u = \{ u_i \}_{i=0}^n \) denote a collection of distinct values over which a piecewise quadratic Bezier function will be defined. The \( j \)'th segment will be defined over \([u_j, u_{j+1})\) as

\[
\gamma_j(t) = \sum_{i=0}^{2} R_{j,i} \theta_{i,2} \left( \frac{t-u_j}{u_{j+1}-u_j} \right),
\]

where \( R_{j,1} = P_{j+1} \) and for \( C^0 \) continuity, \( R_{j,2} = R_{j+1,0} \).
Constructive Piecewise Curves (Cont.)

$C^1$ continuity constraint at $t = u_{j+1}$ yields,

$$\gamma_j'(u_{j+1}) = \gamma_{j+1}'(u_{j+1})$$

or,

$$\frac{2}{u_{j+1} - u_j} (R_{j,2} - R_{j,1}) = \frac{2}{u_{j+2} - u_{j+1}} (R_{j+1,1} - R_{j+1,0}).$$
Constructive Piecewise Curves (Cont.)

Let \( Q_{2(j+1)} = R_{j,2} = R_{j+1,0} \) and \( Q_{2j+1} = R_{j,1} = P_{j+1} \), for \( j = 0, \ldots, n-3 \):

Then, the last equation may be rewritten as

\[
\frac{2}{u_{j+1} - u_j} (R_{j,2} - R_{j,1}) = \frac{2}{u_{j+2} - u_{j+1}} (R_{j+1,1} - R_{j+1,0})
\]
Constructive Piecewise Curves (Cont.)

\[
\frac{2}{u_{j+1} - u_j} (Q_{2(j+1)} - P_{j+1}) = \frac{2}{u_{j+2} - u_{j+1}} (P_{j+2} - Q_{2(j+1)})
\]

Solving for \( Q_{2(j+1)} \),

\[
Q_{2(j+1)} = \frac{u_{j+2} - u_{j+1}}{u_{j+2} - u_j} P_{j+1} + \frac{u_{j+1} - u_j}{u_{j+2} - u_j} P_{j+2}.
\]

Assuming \( u_i = i \), yields

\[
Q_{2(j+1)} = \frac{1}{2} P_{j+1} + \frac{1}{2} P_{j+2}.
\]
Definition 6.2

A sequence $u = \{ u_i \}_{i=0}^{s}$ of distinct real values is called a **breakpoint sequence**.

An associated sequence of positive integer values, $m = \{ m_i \}_{i=0}^{s}$, one for each element of $u$, is called the **multiplicity vector**.

A nondecreasing sequence of real numbers $t = \{ t_i \}$ such that $m_i = \text{card} \{ j \mid t_j = u_i \}$ and $t_j \leq t_{j+1}$ is called a **knot vector**.
Definition 6.2 (Cont.)

Example:

The knot vector \( t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \} \)

has the breakpoint sequence \( u = \{ 0, 1, 4, 5 \} \) and

the multiplicity vector of \( m = \{ 4, 1, 1, 4 \} \).
Algorithm 6.4 (Recursive B-Spline Alg.)

To define a piecewise polynomial curve of degree $k$, called $\gamma(t)$, we require the domain to be $[t_k, t_{N-k})$, where \( \{ t_i \}_{i=0}^{N} \) is as defined in Definition 6.2.

1. For a given $t_{N-k} > t \geq t_k$, find $J$ such that $t \in [t_J, t_{J+1}).$
2. Define $P_i^{[0]} = P_i$.
3. For $p = 1, \ldots, k$, set
   \[
   P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},
   \]
   $i = J - k + p, \ldots, J$.
4. Then, $\gamma(t) = P_J^{[k]}$. 
Example 6.5

Let \( k = 3 \) and \( t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \} \). Given \( t = 2, J = 4 \) since \( t_4 = 1 \leq 2 < t_5 = 4 \). We now have,

\[
P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},
\]

\( i = J, ..., J - k + p \).

**Question:** What is the domain of the curve?
Example 6.5 (Cont.)

For $k=3$, $p=1$, $i=4$,

$$P_4^{[1]} = \frac{t-t_i}{t_{i+4-p}-t_i} P_i^{[p-1]} + \frac{t_{i+4-p}-t}{t_{i+4-p}-t_i} P_{i-1}^{[p-1]}$$

$$= \frac{t-t_4}{t_{8-1}-t_4} P_4^{[1-1]} + \frac{t_{8-1}-t}{t_{8-1}-t_4} P_3^{[1-1]}$$

$$= \frac{2-1}{5-1} P_4^{[0]} + \frac{5-2}{5-1} P_3^{[0]}$$

$$= \frac{1}{4} P_4^{[0]} + \frac{3}{4} P_3^{[0]}$$
Example 6.5 (Cont.)

For \( k = 3, p = 1, i = 3, \)

\[
P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},
\]

\( i = 4, 3, \ldots, l + p. \)
Example 6.5 (Cont.)

For \( k = 3, p = 1, i = 2, \)

\[
P_2^{[1]} = \frac{t-t_i}{t_{i+4-p}-t_i} P_i^{[p-1]} + \frac{t_{i+4-p}-t}{t_{i+4-p}-t_i} P_{i-1}^{[p-1]},
\]

\[
= \frac{t-t_2}{t_6-t_2} P_2^{[1-1]} + \frac{t_{6-1}-t}{t_{6-1}-t_2} P_1^{[1-1]}
\]

\[
= \frac{2-0}{4-0} P_2^{[0]} + \frac{4-2}{4-0} P_1^{[0]}
\]

\[
= \frac{1}{2} P_2^{[0]} + \frac{1}{2} P_1^{[0]}
\]

\[ t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \} \]

\[ t_0, t_3, t_4, t_5, t_6, t_9 \]
Example 6.5 (Cont.)

For $k = 3, p = 2, i = 4$,

$$P_4^{[2]} = \frac{t - t_i}{t_{i+4} - t_i} P_i^{[p-1]} + \frac{t_{i+4} - t}{t_{i+4} - t_i} P_i^{[p-1]}$$

$$= \frac{t - t_4}{t_8 - t_4} P_4^{[2-1]} + \frac{t_8 - t}{t_8 - t_4} P_3^{[2-1]}$$

$$= \frac{2 - 1}{5 - 1} P_4^{[1]} + \frac{5 - 2}{5 - 1} P_3^{[1]}$$

$$= \frac{1}{4} P_4^{[1]} + \frac{3}{4} P_3^{[1]}$$
Example 6.5 (Cont.)

For $k = 3$, $p = 2$, $i = 3$,

$$\begin{align*}
P^{[2]}_3 &= \frac{t - t_i}{t_{i+4-2} - t_i} P^{[p-1]}_i + \frac{t_{i+4-2} - t_i}{t_{i+4-2} - t_i} P^{[p-1]}_{i-1} \\
&= \frac{t - t_3}{t_{7-2} - t_3} P^{[2-1]}_3 + \frac{t_{7-2} - t_i}{t_{7-2} - t_3} P^{[2-1]}_2 \\
&= \frac{2 - 0}{4 - 0} P^{[1]}_3 + \frac{4 - 2}{4 - 0} P^{[1]}_2 \\
&= \frac{1}{2} P^{[1]}_3 + \frac{1}{2} P^{[1]}_2
\end{align*}$$

$$P_i^{[p]} = \frac{t - t_i}{t_{i+4-p} - t_i} P_i^{[p-1]} + \frac{t_{i+4-p} - t_i}{t_{i+4-p} - t_i} P_{i-1}^{[p-1]},$$

$i = 4, 3, \ldots, 1 + p.$
Example 6.5 (Cont.)

For $k = 3$, $p = 3$, $i = 4$,

$$P_4^{[3]} = \frac{t-t_i}{t_{i+4-p}-t_i} P_4^{[p-1]} + \frac{t_{i+4-p}-t_i}{t_{i+4-p}-t_i} P_3^{[p-1]}$$

$$= \frac{t-t_4}{t_8-3-t_4} P_4^{[3-1]} + \frac{t_{8-3}-t}{t_8-3-t_4} P_3^{[3-1]}$$

$$= \frac{2-1}{4-1} P_4^{[2]} + \frac{4-2}{4-1} P_3^{[2]}$$

$$= \frac{1}{3} P_4^{[2]} + \frac{2}{3} P_3^{[2]}$$

and $\gamma(t) = P_4^{[3]}$. 

$$P_i^{[p]} = \frac{t-t_i}{t_{i+4-p}-t_i} P_i^{[p-1]} + \frac{t_{i+4-p}-t_i}{t_{i+4-p}-t_i} P_{i-1}^{[p-1]},$$

$i = 4, 3, ..., l + p.$

$t = \{ 0, 0, 0, 0, 1, 4, 5, 5, 5, 5 \}$

$t_0 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_9$
Example 6.6

Let $t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$ and $k = 2$. The domain of the curve equals $[0, 4)$.

For $t = [0, 1), J = 2$ (why?) and we now have,

1. $P_i^{[0]} = P_i$, for $i = 0, 1, 2$. 
Example 6.6 (Cont.)

Continuing with \( p = 1, k = 2, t = [0, 1), J = 2, i = 2, \ldots, 1, \)

2. 
\[
P^{[1]}_2 = \frac{t-t_2}{t_2+2-t_2} P^{[0]}_2 + \frac{t_2+2-t}{t_2+2-t_2} P^{[0]}_{2-1},
\]
\[
P^{[1]}_1 = \frac{t-t_1}{t_1+2-t_1} P^{[0]}_1 + \frac{t_1+2-t}{t_1+2-t_1} P^{[0]}_{1-1}.
\]

For \( p = 2, k = 2, t = [0, 1), J = 2, i = 2, \ldots, 2, \)

3. 
\[
P^{[2]}_2 = \frac{t-t_2}{t_2+1-t_2} P^{[1]}_2 + \frac{t_2+1-t}{t_2+1-t_2} P^{[1]}_{2-1}.
\]
Example 6.6 (Cont.)

Thus, for $t = [0, 1)$, the curve $\gamma(t)$ is quadratic,

\[
\gamma(t) = \frac{t - t_2}{t_{2+1} - t_2} \left( \frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_2^{[0]} \right) \\
+ \frac{t_{2+1} - t}{t_{2+1} - t_2} \left( \frac{t - t_1}{t_{1+2} - t_1} P_1^{[0]} + \frac{t_{1+2} - t}{t_{1+2} - t_1} P_1^{[0]} \right).
\]
$t = \{ 0, 0, 0, 1, 3, 4, 4, 4 \}$

$$p_i^{[p]} = \frac{t - t_i}{t_i + k - (p - 1) - t_i} p_i^{[p-1]} + \frac{t_i + k - (p - 1) - t}{t_i + k - (p - 1) - t_i} p_i^{[p-1]}$$

**(Example 6.6 (Cont.))**

Similarly, for $t = [1, 3)$, $J = 3$ (why?) and we now have,

1. $P_i^{[0]} = P_i$, for $i = 1, 2, 3$ and recall that $k = 2$.

2. Continuing with $p = 1$,
   
   $i = 3, \ldots, 2$:

   $$p_3^{[1]} = \frac{t - t_3}{t_3 + 2 - t_3} P_3^{[0]} + \frac{t_3 + 2 - t}{t_3 + 2 - t_3} P_3^{[0]}$$

   $$p_2^{[1]} = \frac{t - t_2}{t_2 + 2 - t_2} P_2^{[0]} + \frac{t_2 + 2 - t}{t_2 + 2 - t_2} P_2^{[0]}$$

3. And $p = 2$,
   
   $i = 3, \ldots, 3$:

   $$p_3^{[2]} = \frac{t - t_3}{t_3 + 1 - t_3} P_3^{[1]} + \frac{t_3 + 1 - t}{t_3 + 1 - t_3} P_3^{[1]}$$
Example 6.6 (Cont.)

Thus, for $t = [1, 3)$, the curve $\gamma(t)$ is quadratic as well,

$$
\gamma(t) = \frac{t - t_3}{t_{3+1} - t_3} \left( \frac{t - t_3}{t_{3+2} - t_3} P_3^{[0]} + \frac{t_{3+2} - t}{t_{3+2} - t_3} P_3^{[0]} \right) + \frac{t_{3+1} - t}{t_{3+1} - t_3} \left( \frac{t - t_2}{t_{2+2} - t_2} P_2^{[0]} + \frac{t_{2+2} - t}{t_{2+2} - t_2} P_2^{[0]} \right).
$$

A similar result could be obtained for $t = [3, 4)$.
The B-spline Blending Functions

We seek blending functions that extend the properties of Bezier curves to piecewise polynomials.

Following the recursive Bezier Basis functions’ scheme we seek functions of degree $k$ that satisfy

$$
\gamma(t) = \sum P_i B_{i,k}(t) = \sum P_i^{[j]} B_{i,k-j}(t) \quad j = 0, \ldots, k.
$$
The B-spline Blending Functions (Cont.)

For a given \( t \), there are only \( k+1 \) values of \( P_i^{[0]} \), \( i = J - k, \ldots, J \) that contribute, so,

\[
\gamma(t) = \sum P_i^{[j]} B_{i,k-j}(t)
\]

\[
= \sum \left( \frac{t-t_i}{t_{i+k-j+1}-t_i} P_i^{[j-1]} + \frac{t_{i+k-j+1} - t}{t_{i+k-j+1}-t_i} P_{i-1}^{[j-1]} \right) B_{i,k-j}(t)
\]

\[
= \sum P_i^{[j-1]} \left( \frac{t-t_i}{t_{i+k-j+1}-t_i} B_{i,k-j}(t) + \frac{t_{i+1+k-j+1} - t}{t_{i+1+k-j+1}-t_{i+1}} B_{i+1,k-j}(t) \right)
\]

\[
= \sum P_i^{[j-1]} B_{i,k-j+1}(t).
\]
The B-spline Blending Functions (Cont.)

Hence, and as we seek \( \gamma(t) = \sum P_i^{[j-1]} B_{i,k-(j-1)}(t) \) with a recursive form, we end up with,

\[
B_{i,k-(j-1)}(t) = \left( \frac{t - t_i}{t_{i+k-j+1} - t_i} B_{i,k-j}(t) + \frac{t_{i+1+k-j+1} - t}{t_{i+1+k-j+1} - t_{i+1}} B_{i+1,k-j}(t) \right)
\]

or letting \( r = k - j \),

\[
B_{i,r+1}(t) = \left( \frac{t - t_i}{t_{i+r+1} - t_i} B_{i,r}(t) + \frac{t_{i+r+2} - t}{t_{i+r+2} - t_{i+1}} B_{i+1,r}(t) \right).
\]
Definition 6.7

Let \( t_0 \leq t_1 \leq \ldots \leq t_N \) be a sequence of real numbers. For \( k = 0, \ldots, N-1 \), and \( i = 0, \ldots, N - (k+1) \), define the \( i \)'th (normalized) B-spline \( B_{i,k} \) of degree \( k \) as

\[
B_{i,0}(t) = \begin{cases} 
1 & \text{for } t_i \leq t < t_{i+1}, \\
0 & \text{otherwise}, 
\end{cases}
\]

and for \( k > 0 \),

\[
B_{i,k}(t) = \begin{cases} 
\frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+1+k} - t}{t_{i+1+k} - t_{i+1}} B_{i+1,k-1}(t), & t_i \leq t < t_{i+k+1}, \\
0 & \text{otherwise}. 
\end{cases}
\]
Consider the case of $k = 0$ and assume $t_i = t_{i+1}$.

**Question:** What is the shape of $B_{i,0}(t)$?

Consider $B_{i,k}(t)$ and assume $t_i = \ldots = t_{i+k+1}$.

**Question:** What is the shape of $B_{i,k}(t)$?

Consider $B_{i,k}(t)$ and assume $t_i = \ldots = t_{i+k} < t_{i+k+1}$.

**Question:** What is the shape of $B_{i,k}(t)$?
Example 6.8

Question: What is the shape of $B_{i,1}(t)$?

$$B_{i,1}(t) = \frac{t-t_i}{t_{i+1}-t_i} B_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} B_{i+1,0}(t)$$

$$= \begin{cases} 
\frac{t-t_i}{t_{i+1}-t_i} & \text{for } t_i \leq t < t_{i+1} \\
\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} & \text{for } t_{i+1} \leq t < t_{i+2}.
\end{cases}$$
Lemma 6.9

If $t \geq t_{i+1+k}$ or $t < t_i$, then $B_{i,k}(t) = 0$; i.e., $B_{i,k}(t)$ can be nonzero only on the interval $[t_i, t_{i+k+1})$.

Proof

By induction.
Lemma 6.10

\[ B_{i,k}(t) > 0 \quad \text{for} \quad t \in (t_i, t_{i+k+1}). \]

Further, \[ B_{i,k}(t_{i+k+1}) = 0. \]

Proof

By definition \[ B_{i,0}(t) > 0 \quad \text{for} \quad t \in (t_i, t_{i+1}). \] Further, \[ B_{i,0}(t_{i+1}) = 0 \] and hence Lemma 6.10 holds for \( k = 0. \)
Lemma 6.10 (Cont.)

Assume Lemma 6.10 holds for degree $k-1$. Consider degree $k$.

**Question:** What is the support of $B_{i,k-1}(t)$? of $B_{i+1,k-1}(t)$? of $B_{i,k}(t)$?

$$B_{i,k}(t) = \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t).$$

$$B_{i,k}(t_{i+k+1}) = \frac{t_{i+k+1} - t_i}{t_{i+k} - t_i} B_{i,k-1}(t_{i+k+1}) + \frac{t_{i+k+1} - t_{i+k+1}}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t_{i+k+1})$$
Definition 6.11

A function which is non-zero only over a finite interval of the real line is called a local function.
Corollary 6.12

If $t \in (t_j, t_{j+1})$, then $B_{i,k}(t) > 0$ for

$$i \in \{ j-k, j-k+1, \ldots, j \}.$$
Theorem 6.13

If $\gamma(t)$ is the curve algorithmically defined in Algorithm 6.4 and $\alpha(t) = \sum P_i B_{i,k}(t)$, where the $B_{i,k}(t)$ are defined in Definition 6.7, then $\gamma(t) = \alpha(t)$.

Proof

$$\alpha(t) = \sum_{m=0}^{n} P_m B_{m,k}(t) = \sum_{m=J-k}^{J} P_m B_{m,k}(t), \quad t \in [t_J, t_{J+1}),$$

since only non zero B-splines contribute.
Theorem 6.13 (Cont.)

\[ \alpha(t) = \sum_{m=0}^{n} P_m B_{m,k}(t) = \sum_{m=J-k}^{J} P_m B_{m,k}(t), \quad t \in [t_j, t_{j+1}), \]

Which, when using the recursive definition equals,

\[
\begin{align*}
= & \sum_{m=J-k}^{J} P_m \left[ \frac{t_{m+k+1} - t}{t_{m+k+1} - t_m} B_{m+1,k-1}(t) + \frac{t - t_m}{t_m - t} B_{m,k-1}(t) \right] \\
= & \sum_{m=J-k}^{J} P_m \frac{t_{m+k+1} - t}{t_{m+k+1} - t_m} B_{m+1,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_m - t} B_{m,k-1}(t) \\
= & \sum_{m=J+1-k}^{J+1} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_m - t} B_{m,k-1}(t).
\end{align*}
\]
Theorem 6.13 (Cont.)

Inspecting the last line,

\[ t \in [t_J, t_{J+1}) \]

\[ \sum_{m=J+1-k}^{J+1} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t), \]

the last term of the first summation and the first term of the last summation equal zero (why?), or,

\[ \alpha(t) = \sum_{m=J+1-k}^{J} P_{m-1} \frac{t_{m+k} - t}{t_{m+k} - t_m} B_{m,k-1}(t) + \sum_{m=J+1-k}^{J} P_m \frac{t - t_m}{t_{m+k} - t_m} B_{m,k-1}(t) \]

\[ = \sum_{m=J+1-k}^{J} \left[ \frac{t_{m+k} - t}{t_{m+k} - t_m} P_{m-1} + \frac{t - t_m}{t_{m+k} - t_m} P_m \right] B_{m,k-1}(t). \]
Theorem 6.13 (Cont.)

Therefore,

\[ \alpha(t) = \sum_{m=J+1-k}^{J} \left[ \frac{t_{m+k} - t}{t_{m+k} - t_m} P_{m-1} + \frac{t - t_m}{t_{m+k} - t_m} P_m \right] B_{m,k-1}(t). \]

Iterating this process \( k-1 \) times yields.

\[ \alpha(t) = \sum_{m=J}^{J} P_m^{[k]} B_{m,0}(t) = P_j^{[k]} = \gamma(t). \]
Corollary 6.14

For $t_i \leq t < t_{i+1}$, the B-spline curve is a convex combination of $P_{i-k}, \ldots, P_i$. This is called the convex hull property.
Corollary 6.16

For a knot vector \( t = \{ t_i \}_{i=0}^N \), and for 
\[
t \in [t_k, t_{N-k}),
\]

\[
\sum_{i=0}^{N-(k+1)} B_{i,k}(t) = 1, \quad \forall k \geq 0.
\]

Further, for \( t < t_k \) or \( t \geq t_{N-k} \), 
\[
\sum_{i=0}^{N-(k+1)} B_{i,k}(t) < 1.
\]
Corollary 6.16 (Cont.)

Proof

Again, by induction on the degree $k$. Let,

\[ j \in \{ k, \ldots, N - (k+1) \}. \]

Question: What do we have for $k = 0$?

Now assume $\sum B_{i,k-1}(t) = 1$. For $t \in [t_j, t_{j+1})$, and $k > 0$,
Corollary 6.16 (Cont.)

\[ \sum_i B_{i,k}(t) = \sum_{i=j-k}^j B_{i,k}(t) \]

\[ = \sum_{i=j-k}^j \left[ \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \right] \]

\[ = \sum_{i=j-k}^j \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \sum_{i=j-k}^j \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \]

\[ = \sum_{i=j-k+1}^j \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \sum_{i=j-k}^{j-1} \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t), \]

as this time the first term of the first summation and the last term of the last summation equal zero.
\[ \sum_{i} B_{i,k}(t) = \sum_{i=j-k+1}^{j} \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \sum_{i=j-k}^{j-1} \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t), \]

**Corollary 6.16 (Cont.)**

\[
\begin{align*}
&= \sum_{i=j-k+1}^{j} \frac{t-t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \sum_{i=j+1-k}^{j} \frac{t_{i+k} - t}{t_{i+k} - t_i} B_{i,k-1}(t) \\
&= \sum_{i=j-k+1}^{j} \left[ \frac{t-t_i}{t_{i+k} - t_i} + \frac{t_{i+k} - t}{t_{i+k} - t_i} \right] B_{i,k-1}(t) \\
&= \sum_{i=j-k+1}^{j} B_{i,k-1}(t) \\
&= \sum_{i=0}^{N-k} B_{i,k-1}(t) \\
&= 1,
\end{align*}
\]

by the induction hypothesis.
Corollary 6.16 (Cont.)

Question: What about for $t < t_k$?

By selecting values $t_{-k} \leq \ldots \leq t_{-1} < t_0$ and considering the augmented knot vector $\{t_{-k}, \ldots, t_{-1}, t_0, t_1, \ldots, t_N\}$, we have for $t \in [t_0, t_{k-1})$:

$$1 = \sum_{i=-k}^{N-(k+1)} B_{i,k}(t) = \sum_{i=-k}^{-1} B_{i,k}(t) + \sum_{i=0}^{N-(k+1)} B_{i,k}(t).$$

Now since the first sum is positive, the second must be less than one.
Theorem 6.17 (Derivatives)

For $B_{i,k}(t)$ a B-spline basis function of degree $k$ defined by the knot vector $t$, when the derivative exists (i.e., between knots and sometimes at knots), it is defined by

$$B'_{i,k}(t) = k \left[ \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} \right].$$
Theorem 6.17 (Cont.)

Proof

Once again, the proof is by induction:

\[ B_{i,0}(t) = \begin{cases} 1 & \text{for } t_i \leq t < t_{i+1} \\ 0 & \text{otherwise,} \end{cases} \]

and hence \( B_{i,0}(t) = 0 \) for all \( t \) but \( t = t_i \) and \( t = t_{i+1} \), where is it discontinuous.
Theorem 6.17 (Cont.)

Then, consider $k = 1$ (assuming $t_i < t_{i+2}$),

$$B_{i,1}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,0}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,0}(t),$$

so for $t \not\in \{t_i, t_{i+1}, t_{i+2}\}$,

$$B'_{i,1}(t) = \frac{1}{t_{i+1} - t_i} B_{i,0}(t) - \frac{1}{t_{i+2} - t_{i+1}} B_{i+1,0}(t)$$

$$+ \frac{t - t_i}{t_{i+1} - t_i} B'_{i,0}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B'_{i+1,0}(t)$$

$$= \frac{1}{t_{i+1} - t_i} B_{i,0}(t) - \frac{1}{t_{i+2} - t_{i+1}} B_{i+1,0}(t),$$

following Theorem 6.17.
Theorem 6.17 (Cont.)

Now assume the theorem holds for degree \( k-1 \), for \( t \not\in \{t_j, \ldots, t_{j+k-1}\} \), and prove it for degree \( k \).

Differentiating \( B_{i,k}(t) \):

\[
B'_{i,k}(t) = \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} + \frac{t - t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t).
\]

We now examine the last two terms only,
Theorem 6.17 (Cont.)

Examining the last two terms only,

\[
B'_{i,k}(t) = k \left[ \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} \right].
\]

\[
= \frac{t-t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1}-t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t)
\]

\[
= \frac{t-t_i}{t_{i+k} - t_i} (k-1) \left[ \frac{1}{t_{i+k-1} - t_i} B_{i,k-2}(t) - \frac{1}{t_{i+k} - t_{i+1}} B_{i+1,k-2}(t) \right]
\]

\[
+ \frac{t_{i+k+1}-t}{t_{i+k+1} - t_{i+1}} (k-1) \left[ \frac{1}{t_{i+k} - t_{i+1}} B_{i+1,k-2}(t) - \frac{1}{t_{i+k+1} - t_{i+2}} B_{i+2,k-2}(t) \right],
\]

by the induction hypothesis for this theorem.
Theorem 6.17 (Cont.)

\[
(k - 1) \left[ \frac{1}{t_{i+k} - t_i} \left( \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-2}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1,k-2}(t) \right) \right] \\

- (k - 1) \left[ \frac{1}{t_{i+k} - t_{i+1}} \left( \frac{t_{i+k} - t}{t_{i+k} - t_i} + \frac{t - t_i}{t_{i+k} - t_{i+1}} - \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} \right) B_{i+1,k-2}(t) \right] \\

= (k - 1) \left[ \frac{1}{t_{i+k} - t_i} B_{i,k-1}(t) \right] \\

- (k - 1) \left[ \frac{1}{t_{i+k+1} - t_{i+1}} \left( \frac{t - t_{i+1}}{t_{i+k+1} - t_{i+1}} B_{i+1,k-2}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+2}} B_{i+2,k-2}(t) \right) \right] \\

= (k - 1) \left[ \frac{1}{t_{i+k} - t_i} B_{i,k-1}(t) - \frac{1}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \right]
\]
Theorem 6.17 (Cont.)

Going back to the original differentiation of $B_{i,k}(t)$:

$$B'_{i,k}(t) = \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}$$

$$+ \frac{t - t_i}{t_{i+k} - t_i} B'_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B'_{i+1,k-1}(t)$$

$$= \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} + (k-1) \left[ \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} \right]$$

$$= k \left[ \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} \right],$$

completing the proof.
Theorem 6.18

For B-spline curve \( \gamma(t) = \sum_{i=0}^{n} P_i B_{i,k}(t) \), the \( j \)th derivative is given by

\[
\frac{d^j}{dt^j}(\gamma(t)) = \sum_{i=j}^{n} Q_{j,i} B_{i,k-j}(t)
\]

where

\[
Q_{j,i} = \begin{cases} 
P_i, & \text{for } j = 0, \\
(k-j+1) \frac{Q_{j-1,i} - Q_{j-1,i-1}}{t_{i+k-j+1} - t_i}, & \text{for } j > 0.
\end{cases}
\]
Theorem 6.18 (Cont.)

Proof

For \( j = 0 \), the theorem clearly holds.

Assume the theorem holds for all lower degree derivatives and prove the theorem for the \( j \)th derivative.

Then,
Theorem 6.18 (Cont.)

\[ \frac{d^j}{dt^j} (\gamma(t)) = \frac{d}{dt} \left( \frac{d^{j-1}}{dt^{j-1}} (\gamma(t)) \right) \]

\[ = \frac{d}{dt} \left( \sum Q_{j-1,i} B_{i,k-j+2-1}(t) \right) \]

\[ = \sum Q_{j-1,i} \frac{d}{dt} B_{i,k-j+2-1}(t) \]

\[ = \sum Q_{j-1,i} (k - j + 1) \left( \frac{B_{i,k-j}}{t_{i+k-j+1} - t_i} - \frac{B_{i+1,k-j}}{t_{i+k-j+2} - t_{i+1}} \right), \]

which follows from Theorem 6.17. This proof is completed by expanding the summation and regrouping.
Corollary 6.21 (Integrals)

\[
\int_{-\infty}^{t} B_{i,k}(u) du = \begin{cases} 
0, & t < t_i \\
\sum_{j=i}^{n} \frac{t_{i+1+k} - t_i}{k+1} B_{j,k+1}(t), & t_i \leq t < t_{n+1} \\
\frac{t_{i+1+k} - t_i}{k+1}, & t_{n+1} \leq t
\end{cases}
\]
Corollary 6.21 (Integrals, Cont.)

Proof

By integrating the derivative equation of

\[
B_{i,k}'(u) = k \left( \frac{B_{i,k-1}(u)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(u)}{t_{i+k+1} - t_{i+1}} \right)
\]

And rearranging:

\[
\frac{1}{t_{i+k} - t_i} \int B_{i,k-1}(u) = \frac{B_{i,k}(u)}{k} + \frac{1}{t_{i+k+1} - t_{i+1}} \int B_{i+1,k-1}(u)
\]
Corollary 6.21 (Integrals – Cont.)

Or \[
\int B_{i,k}(u) = \frac{(t_{i+k+1} - t_i)B_{i,k+1}(u)}{k+1} + \frac{t_{i+k+1} - t_i}{t_{i+k+2} - t_{i+1}} \int B_{i+1,k}(u).
\]

For the last non-zero interval \((i = n, B_{n+1,k}(t) = 0)\), we have,

\[
\int B_{n,k}(u) = \frac{(t_{i+k+1} - t_i)B_{n,k+1}(u)}{k+1}.
\]

Or

\[
\int B_{i,k}(u) = \frac{1}{k+1} \sum_{j=i}^{n} (t_{i+k+1} - t_i)B_{j,k+1}(u) = \frac{t_{i+k+1} - t_i}{k+1} \sum_{j=i}^{n} B_{j,k+1}(u).
\]
Corollary 6.22 (Integrals of Curves)

The integral of a curve equals,

\[ \int B_{i,k}(u) = \frac{t_{i+k+1} - t_i}{k+1} \sum_{j=i}^{n} B_{j,k+1}(u). \]

\[ \int \sum_{i=0}^{n} P_i B_{i,k}(u) du = \sum_{j=0}^{n} Q_j B_{j,k+1}(t), \]

where

\[ Q_j = \sum_{i=0}^{j} \frac{(t_{i+k+1} - t_i)}{k+1} P_i. \]
Corollary 6.22 (Integrals of Curves)

Proof (recall $\sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \cdot = \sum_{i=0}^{n} \sum_{j=0}^{n} \cdot$):

$$\int \sum_{i=0}^{n} \sum_{j=0}^{n} P_i B_{i,k}(u) du = \sum_{i=0}^{n} P_i \int B_{i,k}(u) du$$

$$= \sum_{i=0}^{n} P_i \frac{(t_{i+k+1} - t_i)}{k+1} \sum_{j=i}^{n} B_{j,k+1}(t)$$

$$= \frac{1}{k+1} \sum_{j=0}^{n} \left( \sum_{i=0}^{j} (t_{i+k+1} - t_i) P_i \right) B_{j,k+1}(t).$$
Definition 6.23 (Piecewise Polynomials)

If $u$ and $m$ are defined as in Definition 6.2, then

$$PP_{k,u,m} = \{ f(t) \mid (u_i, u_{i+1}) \text{ is a polynomial of degree less than or equal to } k \text{ and } f \in C^{(k-m_i)} \text{ at } u_i \}.$$ 

$PP_{k,u,m}$ is the collection of all piecewise polynomials of degree $k$ with continuity $k-m_i$ at $u_i$.

In other words, $PP_{k,u,m}$ is a vector space.
Theorem 6.24

If \( \{t_j\} \) is a knot sequence developed from \( \{u_j\} \) and \( \{m_j\} \), as in Definition 6.2, then \( B_{i,k} \in PP_{k,u,m} \).

Proof

For \( k = 0 \), \( B_{i,0}(t) \) is continuous at neither \( t_i \) nor \( t_{i+1} \) and Theorem 6.24 holds, having continuity \( k-m_i=0-1=-1 \) or continuity \( C^{(-1)} \). Note \( C^{(j)}, j > 1 \) denotes \( C^{(-1)} \).
Theorem 6.24 (Cont.)

If \( t_i = \ldots = t_{i+k+1} \), then \( B_{i,k}(t) \equiv 0 \) and the theorem is true. Otherwise, the proof is by induction on the order. If

\[
B_{i+k+1}(B_{i,k-1}(t) > 0) \quad \text{and} \quad t_{i+1} < t_{i+k+1} \quad (B_{i+1,k-1}(t) > 0),
\]

\[
B'_{i,k}(t) = (k-1) \left[ \frac{B_{i,k-1}(t)}{t_{i+k} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}} \right],
\]

or a linear combination of \( C^{(k-1-m_i)} \) functions so \( B_{i,k}(t) \), as the integral, is of class \( C^{(k-m_i)} \).
Corollary 6.25

Given knot vector $t$ that has distinct increasing values $u = \{ u_i \}_{i=0}^s$, each repeated with multiplicity $m = \{ m_i \}_{i=0}^s$, then $B_{i,k}(t)$ is contained in $C^{(k-m_p)}$ at $u_p$. 
Theorem 6.26

\[ B_{i,k}(t) \text{ is unimodal. That is, for } t \in (t_i, t_{i+k+1}), \text{ there exists one extremal point.} \]

Proof

If \( t_{i+1} = t_{i+k} \),

\[
B'_{i,k}(t) = \begin{cases} 
  k \frac{B_{i,k-1}(t)}{t_{i+k} - t_i}, & t \in (t_i, t_{i+1}) \\
  -k \frac{B_{i+1,k-1}(t)}{t_{i+k+1} - t_{i+1}}, & t \in (t_{i+1}, t_{i+k+1}).
\end{cases}
\]
Theorem 6.26 (Cont.)

Both $B_{i,k-1}(t)$ and $B_{i+1,k-1}(t)$ are not zeros in their open and disjoint domains.

**Question:** What are the extremals points of $B_{i,k}(t)$?

If, in addition, $t_i = t_{i+1}$, then $B_{i,k-1}(t) \equiv 0$, and $B_{i,k}(t)$ is decreasing. Similarly, if $t_{i+1} = t_{i+k+1}$, $B_{i,k}(t)$ is increasing. Finally, if $t_i < t_{i+1} = t_{i+k} < t_{i+k+1}$, $B_{i,k}(t)$ is increasing for $t < t_{i+1}$, and $B_{i,k}(t)$ is decreasing for $t > t_{i+k}$.
Theorem 6.26 (Cont.)

Now, if \( t_{i+1} < t_{i+k} \), \( B_i',k(t) \) is continuous for \( t \in (t_i, t_{i+k+1}) \), and the extremal points of \( B_i,k(t) \) can occur at the zeros of \( B_i',k(t) \) and nowhere else (we consider an open domain).

Because \( B_{i,k}(t_i) = B_{i,k}(t_{i+k+1}) = 0 \), and using the mean value’s theorem, there exists at least one extremal point for \( B_i',k(t) \). Using properties of roots of spline curves, one can show that this extremal point is unique.
Theorem 6.29

If $u = \{a, b\}$ and $m = \{k+1, k+1\}$, let the knot vector be $t = \{ t_0 = t_1 = \ldots = t_k < t_{k+1} = \ldots = t_{2k+1} \}$, where $t_0 = a$ and $t_{k+1} = b$.

Then, for $t \in [a, b]$, $p = 1, \ldots, k$ and $i = 0, \ldots, p$,

$$B_{i+k-p, p}(t) = \theta_{i,p}\left(\frac{t-a}{b-a}\right),$$

showing that the Bernstein/Bezier blending functions are a special B-spline case.
Theorem 6.29 (Cont.)

Proof:

If $p = 0$, $B_{i+k,0}(t)$ has the support of $[t_{i+k}, t_{i+k+1}]$. Hence,

$$B_{k,0}(t) = 1 = \theta_{0,0}\left(\frac{t-a}{b-a}\right),$$

Or the case of $p = 0$ is proved ($t_{i+k} < t_{i+k+1}$ for $i=0$ only).

We now only consider the general case of $0 < i < p$ (the other cases are left as an exercise):
Theorem 6.29 (Cont.)

\[ t = \{ t_0 = t_1 = \ldots = t_k = a < b = t_{k+1} = \ldots = t_{2k+1} \} \]

\[ t_0 = t_1 = \ldots = t_k = a, \]
\[ t_{k+1} = \ldots = t_{2k+1} = b. \]

\[ i = 1, \ldots, p-1. \]

\[ B_{i+k-p, p}(t) \]
\[ = \frac{t-t_{i+k-p}}{t_{i+k-p} - t_{i+k-p}} B_{i+k-p, p-1}(t) + \frac{t_{i+k+1}-t}{t_{i+k+1} - t_{i+k+1-p}} B_{i+k+1-p, p-1}(t) \]
\[ = \frac{t-a}{b-a} \theta_{i-1, p-1}\left(\frac{t-a}{b-a}\right) + \frac{b-t}{b-a} \theta_{i, p-1}\left(\frac{t-a}{b-a}\right) \]
\[ = \theta_{i, p}\left(\frac{t-a}{b-a}\right). \]

by the induction hypothesis

Question: What is going on for \( i = 0? i = p? \)