Computer Aided Geometric Design

Linear Spaces of B-splines

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based on a book by Elaine, Riesenfeld, & Elber

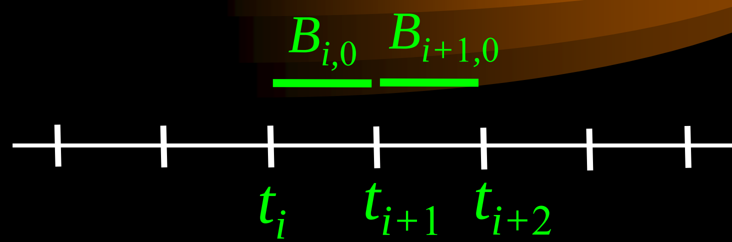
Lemma 7.1

Consider $\boldsymbol{S}_{k,\boldsymbol{t}} = \{\Sigma P_i B_{i,k}(t) : P_i \in \boldsymbol{R}^q\}$, the collection of all linear combination of the modified normalized B-splines of degree k over the knot vector $\boldsymbol{t}$ with coefficients in $\boldsymbol{R}^q$.

$\boldsymbol{S}_{k,\boldsymbol{t}} = \{\Sigma P_i B_{i,k}(t) : P_i \in \boldsymbol{R}^q\}$ is a vector space.

Uniform Floating B-spline Curves
(Section 7.1)

Let ***t*** be defined so that



$t_i = t_0 + ih$, for some real number h.

Without loss of generality, let $h = 1$ and let $t_0 = 0$.

Thus, $t_i = i$. Now,

$$B_{i+1,0}(t) = \begin{cases} 1 & \text{for } t \in [i+1, i+2) \\ 0 & \textit{otherwise} \end{cases},$$

and $B_{i+1,0}(t) = B_{i,0}(t-1)$ for all i.

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+1+k} - t}{t_{i+1+k} - t_{i+1}} B_{i+1,k-1}(t)$$

Uniform Floating B-spline Curves

Hence $B_{i+1,0}$ is just a translation of $B_{i,0}$. Assume $B_{j+1,k-1}(t) = B_{j,k-1}(t-1)$ for all t and all j. Then,

$$B_{i+1,k}(t)$$

$$= \frac{t-(i+1)}{(i+1+k)-(i+1)} B_{i+1,k-1}(t) + \frac{((i+1)+1+k)-t}{((i+1)+1+k)-((i+1)+1)} B_{(i+1)+1,k-1}(t)$$

$$= \frac{(t-1)-i}{(i+k)-i} B_{i+1,k-1}(t) + \frac{(i+k+1)-(t-1)}{(i+k+1)-(i+1)} B_{(i+1)+1,k-1}(t)$$

$$= \frac{(t-1)-i}{(i+k)-i} B_{i,k-1}(t-1) + \frac{(i+k+1)-(t-1)}{(i+k+1)-(i+1)} B_{i+1,k-1}(t-1)$$

$$= B_{i,k}(t-1).$$

Theorem 7.4

Hence, We have shown that:

For $\boldsymbol{t}$ defined so that $t_i = i$,

$$B_{i+1,k}(t) = B_{i,k}(t-1),$$

for all t and k.

Corollary 7.5

For $\boldsymbol{t}$ defined so that $t_i = i$, all B-splines of degree k are translations of each other, i.e.,

$$B_{i+j,k}(t) = B_{i,k}(t-j),$$

for all t, and for all i, j, and k for which the B-splines are defined.

Theorem 7.7

Suppose $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are knot vectors, with lengths $n + k + 2$ and $m + k + 2$, respectively, $m > n \geq k$.

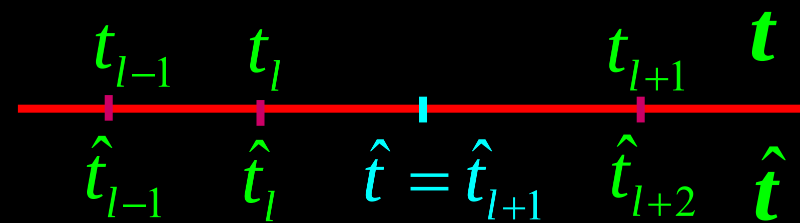
If $\boldsymbol{t}$ contains the elements of $\boldsymbol{\tau}$ with the same or greater multiplicity, then $\boldsymbol{S}_{k,\boldsymbol{\tau}} \subset \boldsymbol{S}_{k,\boldsymbol{t}}$.

Boehm's Knot Insertion (Similar to Lemma 7.8)

(based on "Recursive proof of Boehm's knot insertion technique", by Barry and Goldman, CAD, vol 20, no 4. May 1988)

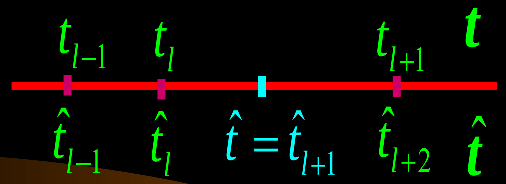
Given an original knot vector, $\boldsymbol{t}$, and a new knot $\hat{t}$, $t_l < \hat{t} \le t_{l+1}$ which produces a refined knot vector, $\hat{\boldsymbol{t}}$,

$$\hat{\boldsymbol{t}} = \begin{cases} \hat{t}_i = t_i, & i \le l \\ \hat{t}_{i+1} = \hat{t}, & \\ \hat{t}_i = t_{i-1}, & i \ge l+2, \end{cases}$$



denote by $B_{i,k}(t)$ and $\hat{B}_{i,k}(t)$ the corresponding B-splines over these two knot vectors, respectively.

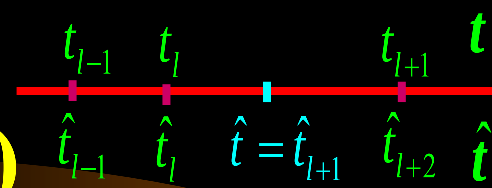
Boehm's Knot Insertion (Cont.)



Then, the B-spline curve $F(t) = \sum_i A_i B_{i,k}(t)$ can be written in terms of the basis $\hat{B}_{i,k}(t)$ as $F(t) = \sum_i \hat{A}_i \hat{B}_{i,k}(t)$,

where

$$\hat{A}_i = \begin{cases} A_i, & i \le l-k \\ \dfrac{\hat{t} - \hat{t}_i}{\hat{t}_{i+k+1} - \hat{t}_i} A_i + \dfrac{\hat{t}_{i+k+1} - \hat{t}}{\hat{t}_{i+k+1} - \hat{t}_i} A_{i-1}, & l-k+1 \le i \le l \\ A_{i-1}, & i \ge l+1. \end{cases}$$



Boehm's Knot Insertion (Cont.)

The result on A_i is a direct consequence of the following recursive identity on the basis functions,

$$B_{i,k}(t) = \begin{cases} \hat{B}_{i,k}(t), & i \leq l-k-1 \\ \dfrac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\hat{B}_{i,k}(t) + \dfrac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\hat{B}_{i+1,k}(t), & l-k \leq i \leq l \\ \hat{B}_{i+1,k}(t), & i \geq l+1. \end{cases} \quad (*)$$

Boehm's Knot Insertion (Cont.)

We prove this (*) relation by induction. The

recursive relation of the B-spline functions is defined as,

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t),$$

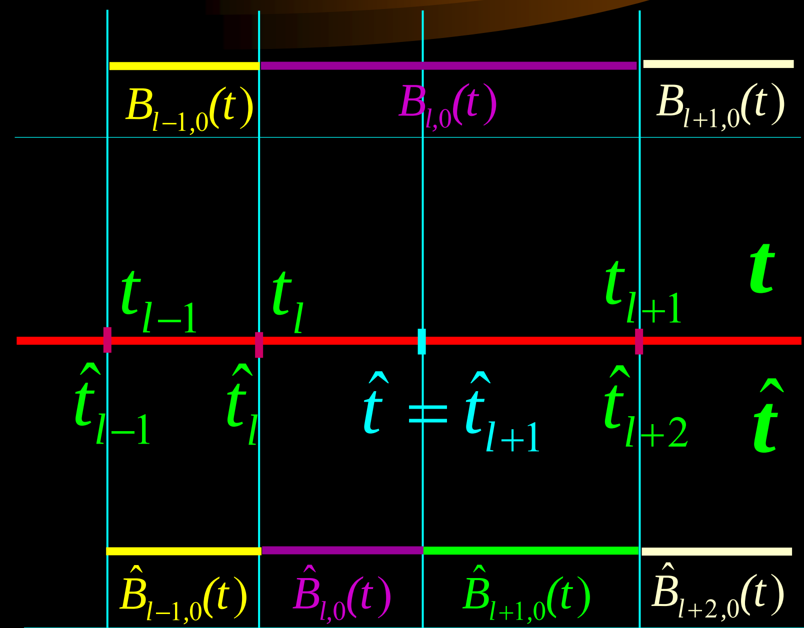
with the initial conditions of $B_{i,0}(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$

$$B_{i,k}(t) = \frac{\hat{t} - \hat{t}_i}{\hat{t}_{i+k+1} - \hat{t}_i} \hat{B}_{i,k}(t) + \frac{\hat{t}_{i+k+2} - \hat{t}}{\hat{t}_{i+k+2} - \hat{t}_{i+1}} \hat{B}_{i+1,k}(t), \quad l-k \le i \le l$$

Boehm's Knot Insertion (Cont.)

For $k = 0$ we have

$$B_{i,0}(t) = \begin{cases} \hat{B}_{i,0}(t), & i \le l-1 \\ \dfrac{\hat{t} - \hat{t}_i}{\hat{t}_{i+1} - \hat{t}_i} \hat{B}_{i,0}(t) + \dfrac{\hat{t}_{i+2} - \hat{t}}{\hat{t}_{i+2} - \hat{t}_{i+1}} \hat{B}_{i+1,0}(t), & l \le i \le l \\ \hat{B}_{i+1,0}(t), & i \ge l+1. \end{cases}$$



(Note that $\hat{t} = \hat{t}_{i+1}$ for $l = i$.)

$$B_{i,k}(t) = \frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\hat{B}_{i,k}(t) + \frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\hat{B}_{i+1,k}(t), \quad l-k \le i \le l$$

Boehm's Knot Insertion (Cont.)

Assume (*) holds for k-1. We will now show that (*) holds for k,

$$B_{i,k}(t) = \frac{t-t_i}{t_{i+k}-t_i}B_{i,k-1}(t) + \frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}B_{i+1,k-1}(t)$$

$$= \frac{t-t_i}{t_{i+k}-t_i}\left[\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t) + \frac{\hat{t}_{i+k+1}-\hat{t}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right] +$$

$$\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}\left[\frac{\hat{t}-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t) + \frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right].$$

Boehm's Knot Insertion (Cont.)

Renaming elements in t as elements of $\hat{t}$ yields,

$$B_{i,k}(t)=\frac{t-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\left[\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t)+\frac{\hat{t}_{i+k+1}-\hat{t}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right]+$$

$$\frac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\left[\frac{\hat{t}-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)+\frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right].$$

and by adding and subtracting

$$\left[\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\frac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}+\frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\frac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\right]\hat{B}_{i+i,k-1}(t),$$

we have,

$$B_{i,k}(t) = \frac{t-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\left[\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t) + \frac{\hat{t}_{i+k+1}-\hat{t}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right] + \left[\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\frac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}} + \frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\frac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\right]\hat{B}_{i+i,k-1}(t),$$

$$\frac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\left[\frac{\hat{t}-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t) + \frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right].$$

Boehm's Knot Insertion (Cont.)

$$\begin{aligned}
B_{i,k}(t) = {} & \frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\left[\frac{t-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t) + \frac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right] \\
& + \frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\left[\frac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t) + \frac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right] \\
& + \left[\begin{array}{l} -\dfrac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\dfrac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}} - \dfrac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\dfrac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}} \\ +\dfrac{t-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\dfrac{\hat{t}_{i+k+1}-\hat{t}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}} + \dfrac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\dfrac{\hat{t}-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}} \end{array}\right]\hat{B}_{i+1,k-1}(t).
\end{aligned}$$

Question: What is the value of the expression in the last set of brackets?

$$B_{i,k}(t)=\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\left[\frac{t-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t)+\frac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right]$$

$$+\frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\left[\frac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)+\frac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right].$$

Boehm's Knot Insertion (Cont.)

Then,

$$B_{i,k}(t)=\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\left[\frac{t-\hat{t}_i}{\hat{t}_{i+k}-\hat{t}_i}\hat{B}_{i,k-1}(t)+\frac{\hat{t}_{i+k+1}-t}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)\right]$$

$$+\frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\left[\frac{t-\hat{t}_{i+1}}{\hat{t}_{i+k+1}-\hat{t}_{i+1}}\hat{B}_{i+1,k-1}(t)+\frac{\hat{t}_{i+k+2}-t}{\hat{t}_{i+k+2}-\hat{t}_{i+2}}\hat{B}_{i+2,k-1}(t)\right]$$

$$=\frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+k+1}-\hat{t}_i}\hat{B}_{i,k}(t)+\frac{\hat{t}_{i+k+2}-\hat{t}}{\hat{t}_{i+k+2}-\hat{t}_{i+1}}\hat{B}_{i+1,k}(t),$$

and we proved (*) for k as well. One also have to consider the limiting cases of $i=l-k+1$ and $i=1$, proofs that follow similar lines.

Definition 7.11

Consider two knot sequences τ and t where τ is a subset of t, that is $t = \{ t_i \} = \tau \cup \eta$ where $\eta = \{ \eta_j \}$ for some non decreasing sequence η. η may contain the same value multiple times.

Then, t is called a refinement of τ.

$$\hat{A}_i = \frac{\hat{t} - \hat{t}_i}{\hat{t}_{i+k+1} - \hat{t}_i} A_i + \frac{\hat{t}_{i+k+1} - \hat{t}}{\hat{t}_{i+k+1} - \hat{t}_i} A_{i-1}, \qquad l - k + 1 \le i \le l.$$

Boehm's Knot Insertion (Cont.)

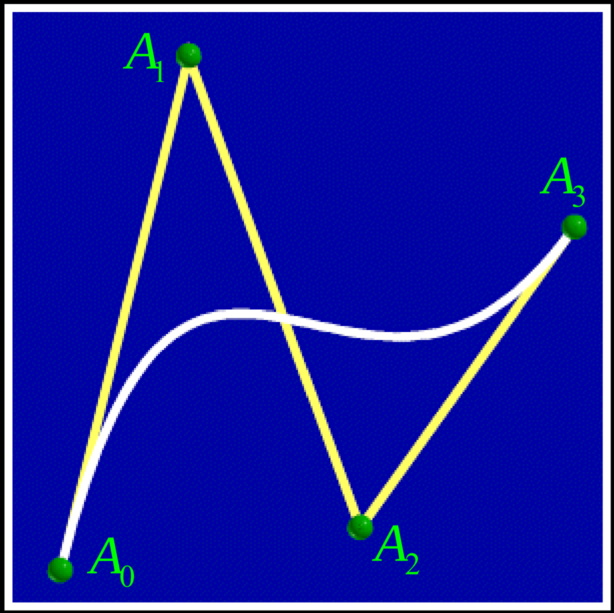
Example: Let $C(t)$ be a cubic ($k = 3$) B-spline curve with the knot sequence of $\boldsymbol{t} = [0,0,0,0,1,1,1,1]$. We now refine the curve $C(t)$ at $\hat{t} = ½$. Hence $l = 3$, yielding

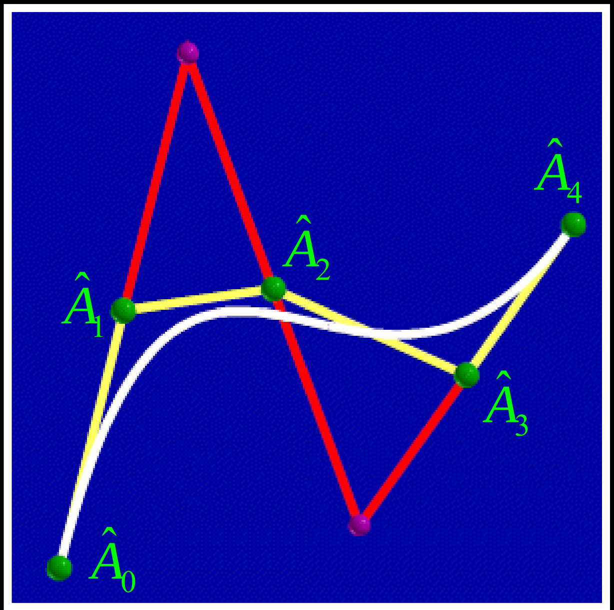
$\hat{\boldsymbol{t}} = [0,0,0,0,½, 1,1,1,1]$ or,

$$\hat{A}_i = \begin{cases} A_i, & i \le 0 \\ \dfrac{\frac{1}{2} - \hat{t}_i}{\hat{t}_{i+4} - \hat{t}_i} A_i + \dfrac{\hat{t}_{i+4} - \frac{1}{2}}{\hat{t}_{i+4} - \hat{t}_i} A_{i-1}, & 1 \le i \le 3 \\ A_{i-1}, & i \ge 4. \end{cases}$$

Boehm's Knot Insertion (Cont.)

Then, we have $\hat{A}_i = \begin{cases} A_i, & i \le 0 \\ \frac{1}{2}A_i + \frac{1}{2}A_{i-1}, & 1 \le i \le 3 \\ A_{i-1}, & i \ge 4. \end{cases}$ or,



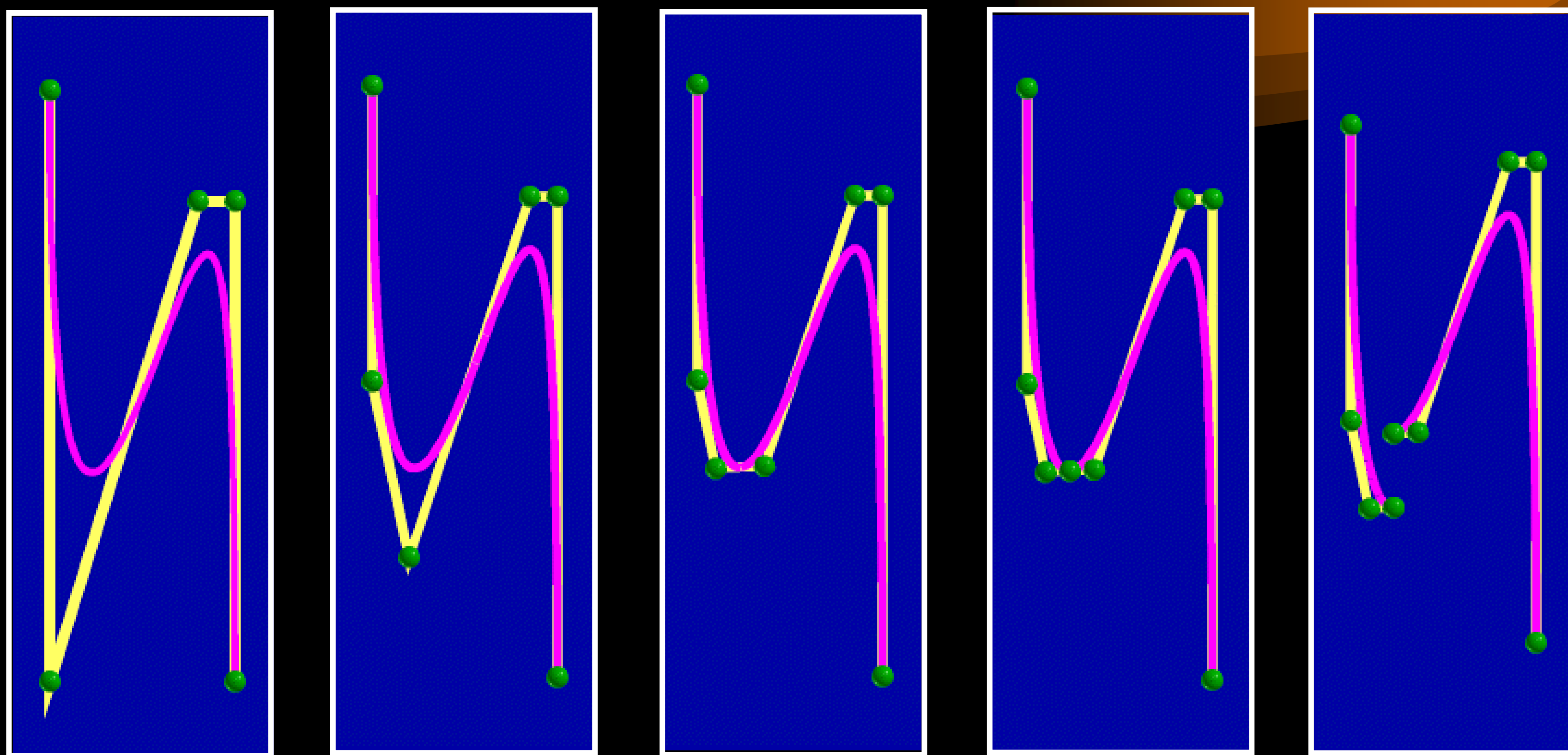


Refinement

Question: For what else can we exploit refinements?

- Subdivision of curves and surfaces.
- Sub-region extraction from curves and surfaces.
- Fast display of curves and surfaces.
- Fine and local control over the shape of curves/surfaces.

Subdivision (Cubic B-spline Curve)



[0,0,0,0,1, 2,2,2,2]

[0,0,0,0,½, 1,2,2,2,2]

[0,0,0,0,½, ½,1,2,2,2,2]

[0,0,0,0,½,½, ½,1,2,2,2,2]

[0,0,0,0,½,½,½,½]
[½,½,½,½,1,2,2,2,2,2]

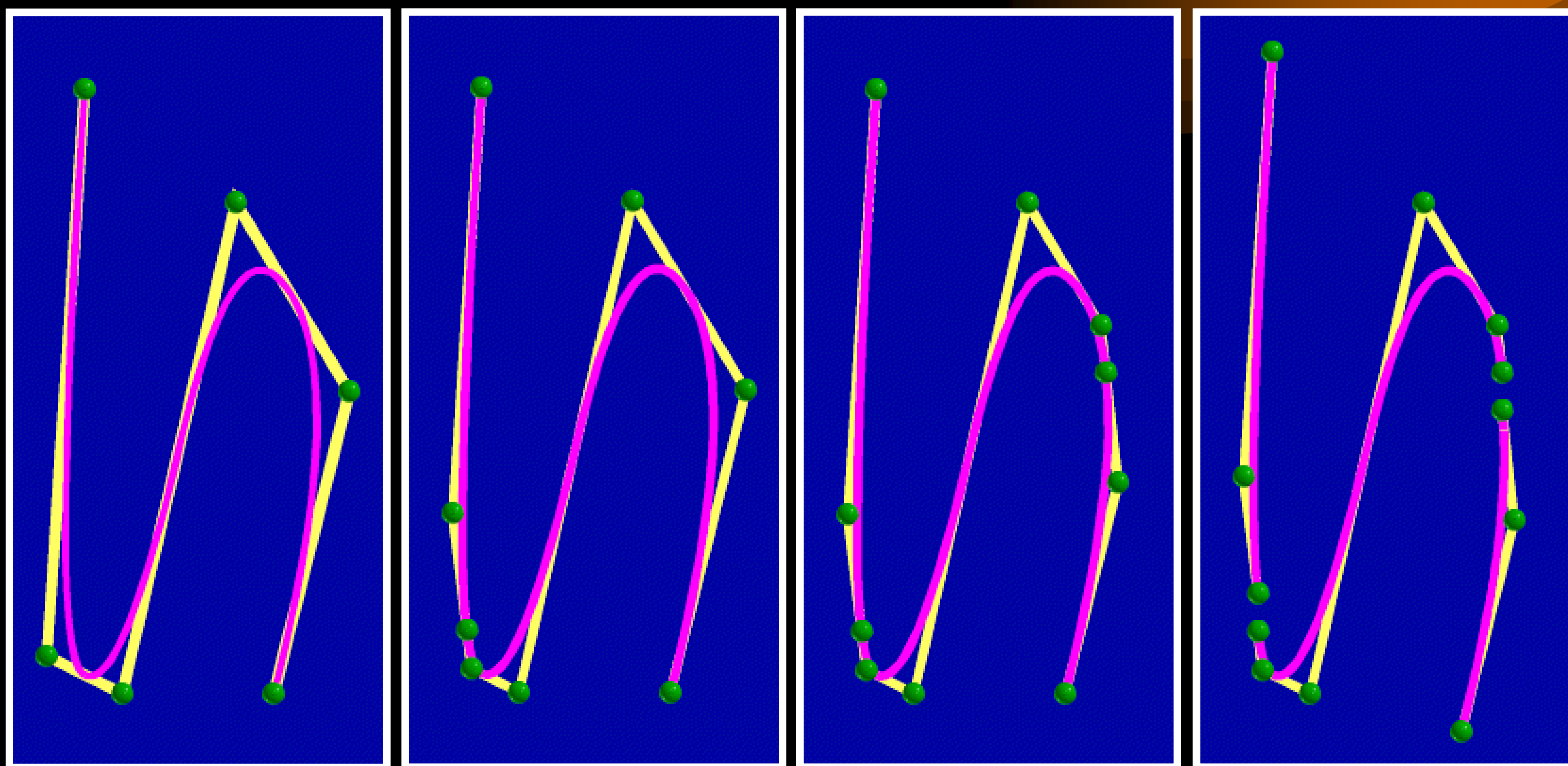
Subdivision

Question: What is the shape of the different B-spline basis functions throughout this refinement process?

Question: What would happen when we insert another (fourth) knot at $t = 0.5$?

Question: How can we extract a sub-region out of a curve? Out of a surface?

Sub-Region Extraction (Quadratic B-spline Curve)



[0,0,0,1,2,
3,4,4,4]

[0,0,0,¾,¾,
1,2,3,4,4,4]

[0,0,0,¾,¾,1,2,
3,3.3,3.3,4,4,4]

[0,0,0,¾,¾,¾]
[¾,¾,¾,1,2,3,3.3,3.3,3.3]
[3.3,3.3,3.3,4,4,4]

Fast Display

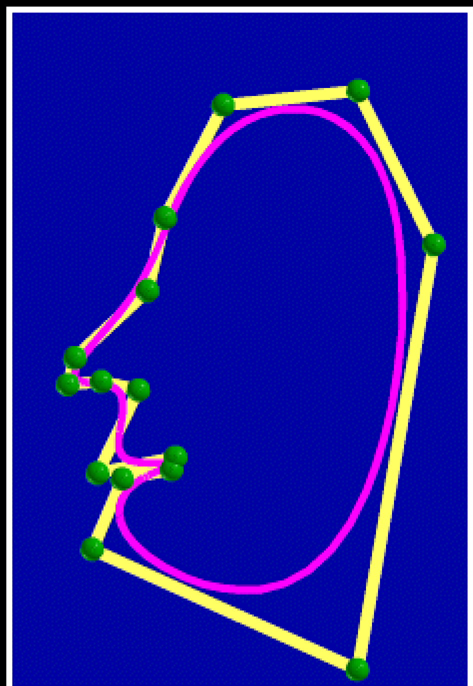
Observation: Given a curve $F(t) = \sum_i A_i B_{i,k}(t)$, the relation between A_i and $\hat{A}_i$ depends solely on the knots.

One can pre-compute a refinement matrix $\boldsymbol{A}$ that refines $F(t)$, with a known knot vector, $\boldsymbol{t} = \{t_i\}$, $i = 1,\ldots,m$, to a new curve with a refined knot vector, $\hat{\boldsymbol{t}} = \{\hat{t}_j\}$, $j = 1,\ldots,n$,

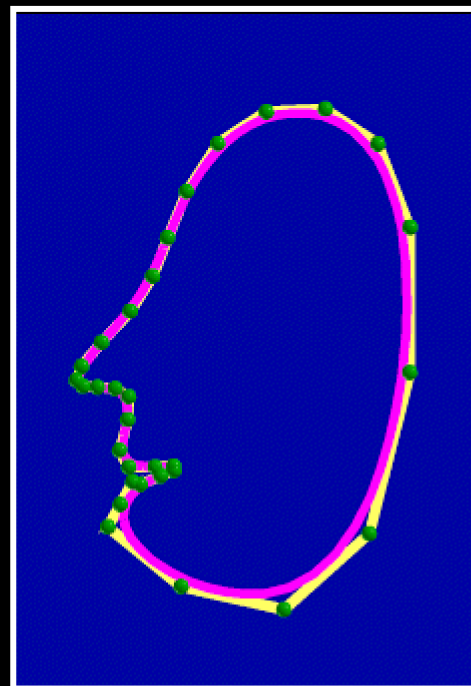
$$\hat{\boldsymbol{t}} = \boldsymbol{A}\boldsymbol{t}.$$

Question: How can we compute $\boldsymbol{A}$? Use $\boldsymbol{A}$?

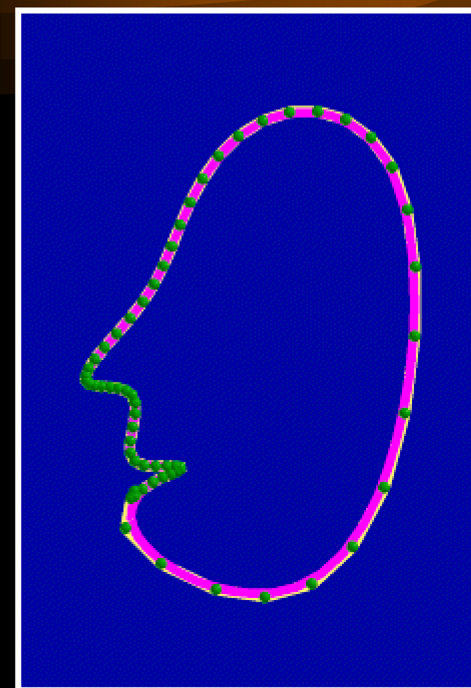
Fast Display
(Quartic Periodic B-spline Curve)



Original Curve



Twice the Control Points



Four Times the Control Points

Fine and Local Control

Question: Given a Bezier curve, how can we manipulate only its first half, leaving the second half unmodified?

Question: Given a B-spline curve, how can we control the domain that is affected by moving a single control point?

Question: Given a B-spline curve, how can we introduce a C^1 discontinuity into the curve? C^0 discontinuity?

Example 7.15

Given a knot vector $\boldsymbol{t}$ and degree k, consider the B-spline curve $\gamma(t)$ in $\boldsymbol{S}_{k,\boldsymbol{t}}$. Clearly monomials are in $\boldsymbol{S}_{k,\boldsymbol{t}}$. Hence,

Question: Find $c_{i,k}$ for the B-spline curve of degree k such that

$$t \equiv \sum_{i=0}^{n} c_{i,k} B_{i,k}(t).$$

Example 7.15

Answer: $$c_{i,k} = t^*_{i,k} = \frac{t_{i+1} + t_{i+2} + \ldots + t_{i+k}}{k}$$

$t^*_{i,k}$ are also know as node values or greville abscissa.

Proof: By induction on the degree k.

Question (Example 7.16): How can one draw an explicit B-spline curve $y = f(x)$ as a parametric B-spline curve?