Computer Aided Geometric Design

Choosing a B-spline Space

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based on a book by Elaine, Riesenfeld, & Elber

Theorem 8.1

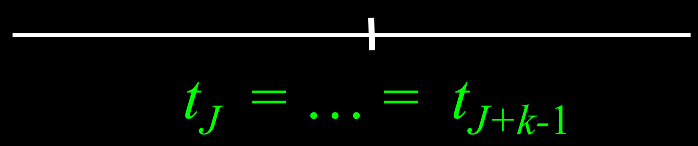
Let $\{t_i\}_{i=0}^{n+k+1}$ be a knot sequence with corresponding B-splines $\{B_{i,k}(t)\}_{i=0}^{n}$ and $\gamma(t)=\sum_{i=0}^{n} P_i B_{i,k}(t),\ t\in[t_k,t_{n+1}]$.

Suppose $t_i < t_{i+k+1}$ for all i so the basis functions of degree k are not degenerate.

If there is a J such that

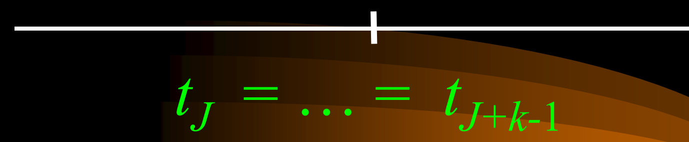
$t_J = \ldots = t_{J+k-1}$, then

there are several possible sub-cases.



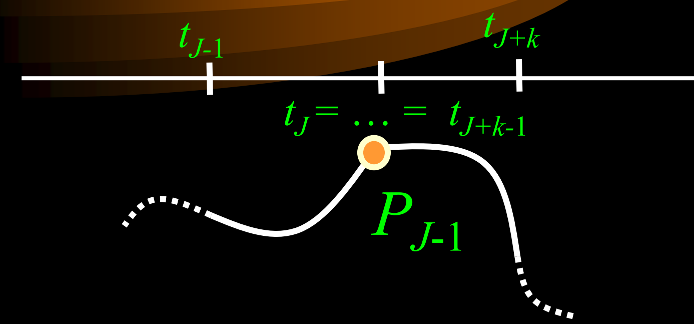
$$\gamma(t) = \sum_{i=0}^{n} P_i B_{i,k}(t)$$

Theorem 8.1 (Cont.)



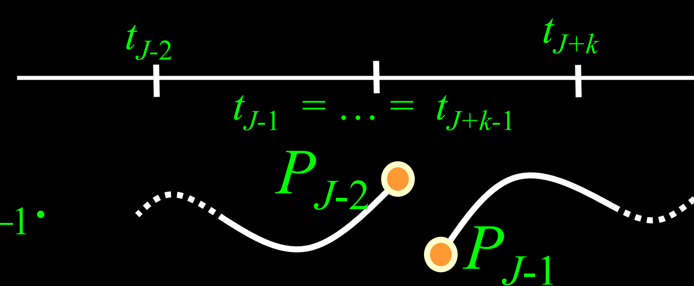
Sub-cases:

If $t_{J\text{-}1} < t_J < t_{J+k}$, the curve is $C^{(0)}$ and interpolates $P_{J\text{-}1}$.



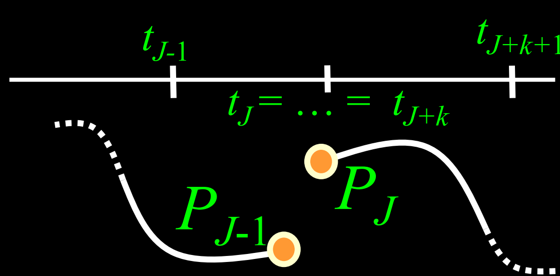
If $t_{J\text{-}1} = t_J$ the curve is $C^{(-1)}$, that is

$$\lim_{t \to t_J^-} \gamma(t) = P_{J-2}, \text{ and } \lim_{t \to t_J^+} \gamma(t) = P_{J-1}.$$



If $t_{J+k} = t_J$ then curve is also $C^{(-1)}$, but

$$\lim_{t \to t_J^-} \gamma(t) = P_{J-1}, \text{ and } \lim_{t \to t_J^+} \gamma(t) = P_J.$$



$$\gamma(t) = \sum_{i=0}^{n} P_i B_{i,k}(t)$$

Theorem 8.1 (Cont.)

$t_J = \ldots = t_{J+k-1}$

Question: What is the shape of the curve, $\gamma(t)$, near t_J if:

- $t_{J-1} < t_J < t_{J+k}$, (the curve is $C^{(0)}$)?
- $t_{J-1} = t_J$, (the curve is $C^{(-1)}$)?
- $t_{J+k} = t_J$, (the curve is $C^{(-1)}$)?

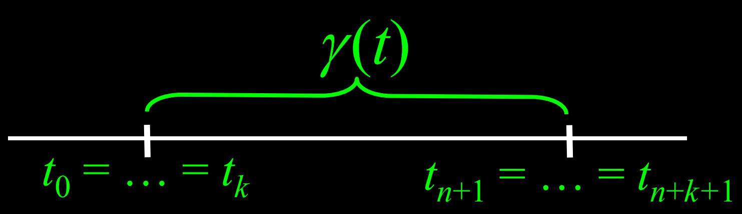
What is the shape of $\gamma'(t)$ at t_J?

Corollary 8.2

For $\mathbf{t} = \{t_i\}_{i=0}^{n+k+1}$ such that $t_0 = \ldots = t_k$, and $t_{n+1} = \ldots = t_{n+k+1}$,

the curve

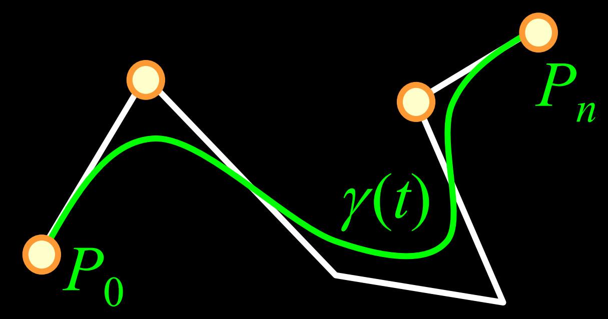
$$\gamma(t) = \sum_{i=0}^{i=n} P_i B_{i,k}(t)$$



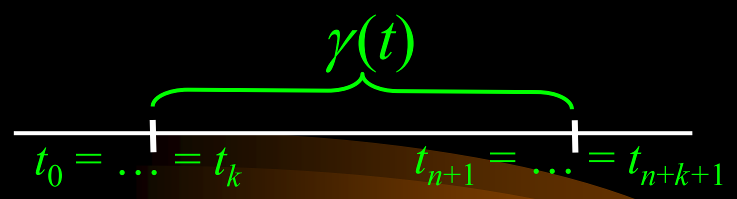
interpolates P_0 and P_n and is

tangent to $(P_1 - P_0)$ and $(P_n - P_{n-1})$

at $t = t_k$ and $t = t_{n+1}$, respectively.



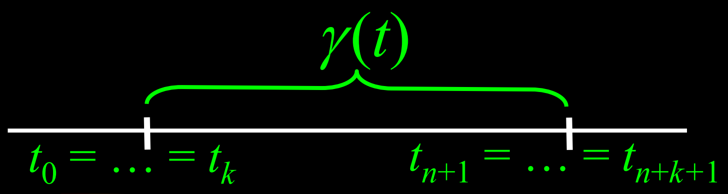
Definition 8.3



The B-spline curve defined in Corollary 8.2 is called the **Open End B-spline** curve.

Question: What is the shape of the B-spline basis functions at the ends of an open-end B-spline curve?

Question: What is the shape of an open-end B-spline curve, with no interior knots?



Open-end B-spline Curve & No Interior Knots

Recall the constructive B-spline algorithm:

- Find J such that $t \in [t_J, t_{J-1})$ and let $P_i^{[0]} = P_i$.
- For $p = 1, \ldots, k$, set

$$P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$$

$$i = J, J-1, \ldots, J-k+p.$$

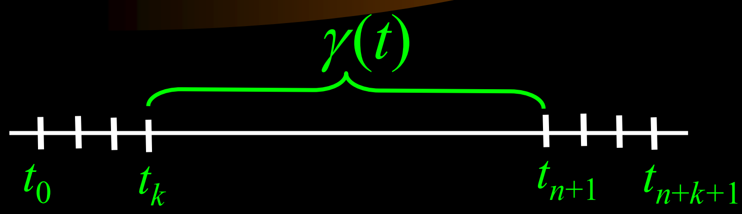
- Then, $\gamma(t) = P_J^{[k]}$.

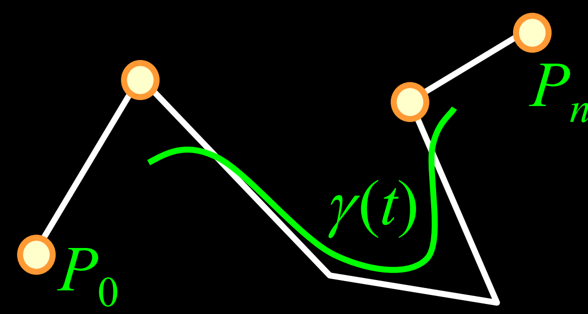
Here $J = k$ or $P_i^{[p]} = \frac{t - t_i}{t_{i+k-(p-1)} - t_i} P_i^{[p-1]} + \frac{t_{i+k-(p-1)} - t}{t_{i+k-(p-1)} - t_i} P_{i-1}^{[p-1]},$

$i = k, k-1, \ldots, p,$ or $i \le k$ and $i+k-(p-1) > k$.

Definition 8.4

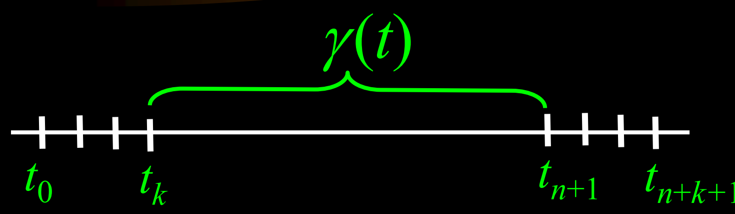
A B-spline curve of order k defined over $[t_k, t_{n+1})$ for which the multiplicities necessary for interpolation do not apply at either end is called **a** **Floating End B-spline** curve.

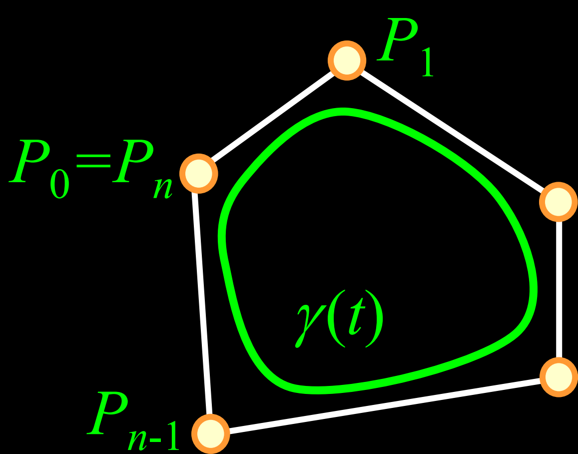




Definition 8.5

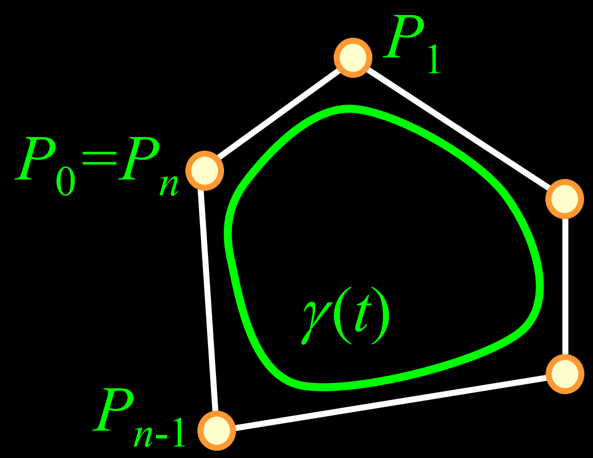
A B-spline curve is called **Periodic** if there exists a real value $d = t_{n+1} - t_k$ such that $\gamma(t) = \gamma(t + d)$, control points $P_i = P_{i \bmod n}$, and basis functions $B_{i,k}(t) = B_{i \bmod n, k}(t)$.





Definition 8.5 (Cont.)

A B-spline periodic curve could be defined with a parametric domain of a unit circle.



A mapping from the real line (non periodic parametric domain) to the unit circle could be made via:

$$Circle(t) = \exp\left(2\pi i \frac{t-t_0}{t_{n+1}-t_0}\right)$$

Definition 8.6

Let $\Delta_i = t_{i+1} - t_i$. If $\Delta_i = \Delta_{i+1}$, for all i, then all B-spline curves using $\boldsymbol{t}$ as a knot vector are called **Uniform Floating** or **Uniform Periodic B-spline** curves, depending on whether the modulus identification is made.

If the end conditions are open, the curve is denoted a **Uniform Open B-spline** curve. All other B-spline curves are called **Non-uniform B-spline** curves.

Questions on Knot Vectors

Question: Given knot vector, $\boldsymbol{t}$, what is the effect of letting $t_i \Leftarrow t_i + c$, for some real number c, on the curve defined over $\boldsymbol{t}$?

Question: Given knot vector, $\boldsymbol{t}$, what is the effect of letting $t_i \Leftarrow t_i c$, for some positive real number c, on the curve defined over $\boldsymbol{t}$?

Definition 8.7 (Rational Curves)

Let $\boldsymbol{t} = \{t_0, \ldots, t_{n+k+1}\}$, $t_i \leq t_{i+1}$, and $t_i < t_{i+k+1}$, for all appropriate i, and let $B_{0,k}(t), \ldots, B_{n,k}(t)$ be the $n + 1$ B-splines defined over that knot vector. For a sequence of coefficients $\{P_0, \ldots, P_n\}$, which can be scalar or vector and a sequence of scalars, $\{w_0, \ldots, w_n\}$, the curve

$$\gamma(t) = \frac{\sum_{i=0}^{n} w_i P_i B_{i,k}(t)}{\sum_{i=0}^{n} w_i B_{i,k}(t)},$$

is called a **rational B-spline curve** of degree k.

Definition 8.7 (Cont.)

The w_i's are sometimes called **homogeneous coordinates** for the P_i's since there is one w_i associated with each P_i . The determining parameters of the rational B-spline curve are the knot vector and the **homogeneous point**

$$H_i = (h_{x,i}, h_{y,i}, h_{z,i}, h_{w,i}) \quad \text{where if } P_i = (x_i, y_i, z_i),$$

$$h_{x,i} = w_i x_i, \quad h_{y,i} = w_i y_i, \quad h_{z,i} = w_i z_i \text{ and } h_{w,i} = w_i .$$

Warning: Some systems use $P_i = (w_i x_i, w_i y_i, w_i z_i)$!

Multiple Knots vs. Multiple Vertices

(Section 8.4)

We already know that **Open End** conditions result in the interpolation of the end points of the control polygon.

Consider a curve with **Uniform Floating End** conditions.

Question: What will be the end point of a curve of degree k, $\gamma(t_k)$, if $P_0 = P_1 = \ldots = P_{k-1}$? Also $P_0 = P_k$?

Question: What is the curvature of the curve, $\kappa(t)$, at $t = [t_k, t_{k+1})$, if $P_0 = P_1 = \ldots = P_{k-1}$? Also $P_0 = P_k$?

Some General Notes

- Symmetry of Bezier/B-spline functions.
- Multi-resolution editing of B-spline functions.
- Summation/difference of B-spline functions.
- Knot removal.
- Degree raising of B-spline functions.
- Product of B-spline functions.
- Composition of B-spline functions.
- Zeros of Bezier/B-spline functions.
- Interrogation of B-spline functions: x/y extrema, adaptive isolines' coverage.