Computer Aided Geometric Design

Conic Sections

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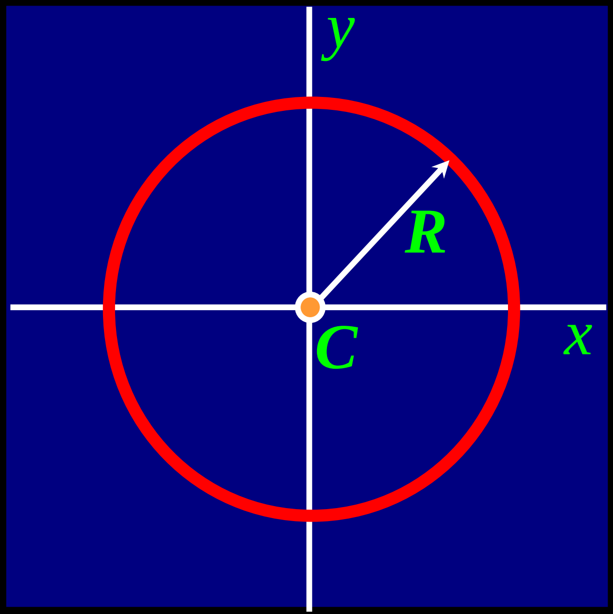
based on a book by Cohen, Riesenfeld, & Elber

Definition 3.1 (The Circle)

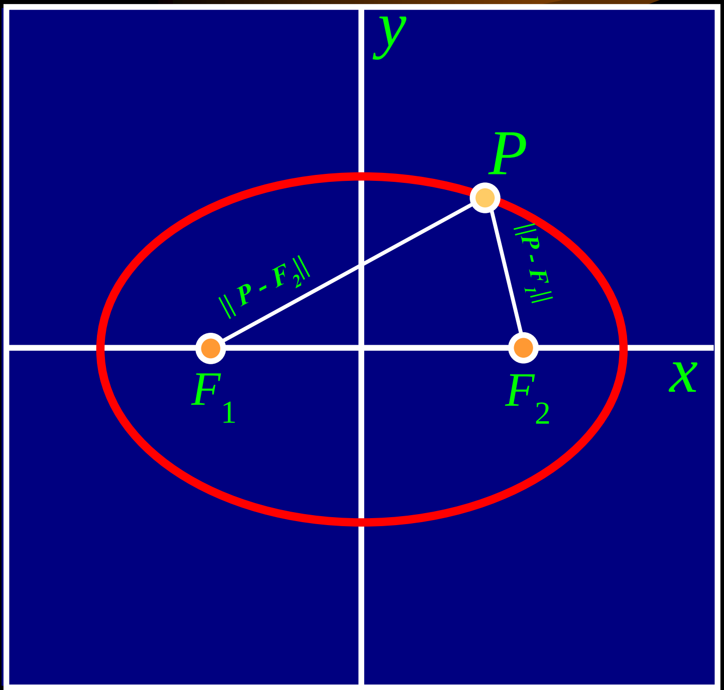
Given a point C in a plane and a number $R \geq 0$, the **circle** with center C and radius R is defined as the set of all points in the plane at distance R from the point C. In set notation we write,

$$\{ P = (x, y) : \| P - C \| = R \}$$



Definition 3.2 (The Ellipse)

Given two points, F_1 and F_2 called the **foci** and a number $K \geq \| F_1 - F_2 \|$, an **ellipse** is defined as the set of all points the sum of whose distances from the foci K. That is,



$$\{ P = (x, y) : \| P - F_1 \| + \| P - F_2 \| = K \} .$$

The Ellipse

The axis containing the foci is called the major axis of the ellipse

and the axis orthogonal to the major axis through the center,

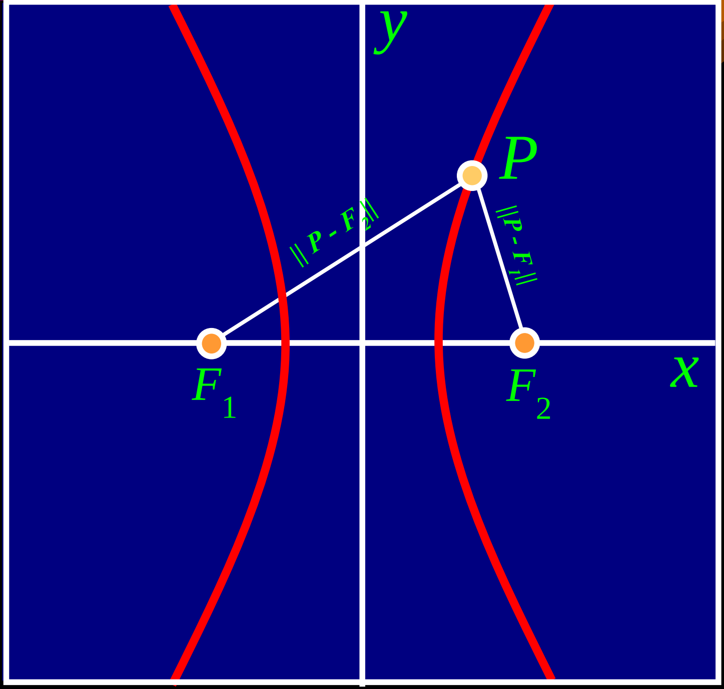
$C = (F_1 + F_2) / 2$, is denoted the minor axis.

If center C is at some location $C = (c_x, c_y)$ the ellipse equals,

$$1 = \frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2}.$$

Definition 3.3 (The Hyperbola)

Given two points, F_1 and F_2 called the **foci** and a number $K \neq 0$, a **hyperbola** is defined as the set of all points the **difference** of whose distances from the foci K. That is,



$$\{ P = (x, y) : \| P - F_1\| - \| P - F_2\| = \pm K \}.$$

Question: What if $K = 0$?

The Hyperbola

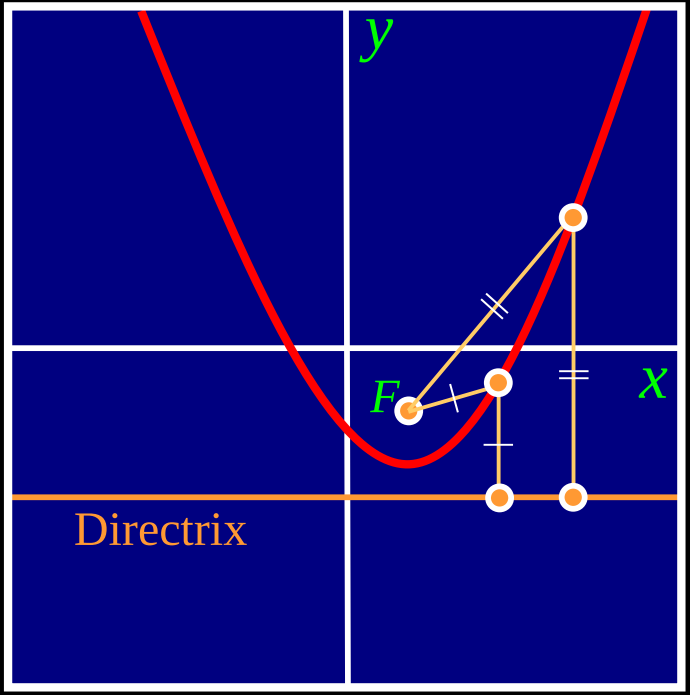
The axis containing the foci is called the major axis of the hyperbola and the axis orthogonal to the major axis through the center, $C = (F_1 + F_2) / 2$, is denoted the minor axis.

If center C is at some location $C = (c_x, c_y)$ the hyperbola equals,

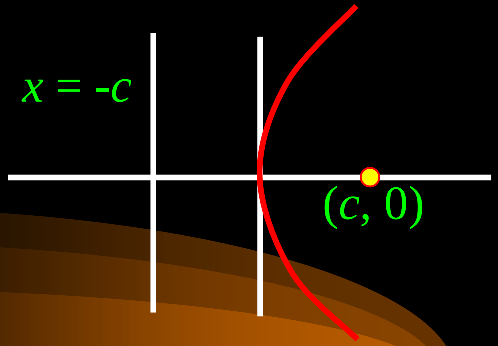
$$1 = \frac{(x - c_x)^2}{a^2} - \frac{(y - c_y)^2}{b^2} \qquad \text{or} \qquad 1 = \frac{(y - c_y)^2}{a^2} - \frac{(x - c_x)^2}{b^2}.$$

Definition 3.4 (The Parabola)

Given a fixed point *F* called the **focus** and a fixed line called the **directrix**, the set of points equidistant (bisector) from *F* and the **directrix** is called a **parabola**.



The Parabola



Let the **directrix** be the line $x = -c$ and assume $F = (c, 0)$. Then,

$$x + c = \sqrt{(x - c)^2 + y^2},$$

and by squaring the expression,

$$\underline{x^2} + 2xc + \underline{\underline{c^2}} = \underline{x^2} - 2xc + \underline{\underline{c^2}} + y^2$$
$$4xc = y^2.$$

Conic Sections

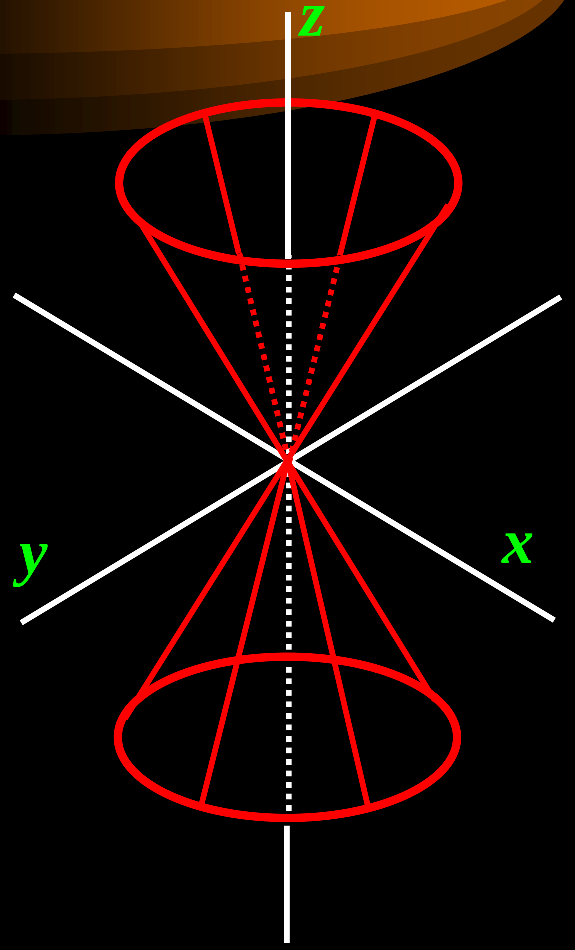
Consider the cone $z^2 = m^2(x^2 + y^2)$,

where m is a real number.

Question: What is the affect of m?

Question: What is the shape of a

Plane-Cone intersection?



Conic Sections

Consider the plane $x = k$. Then,

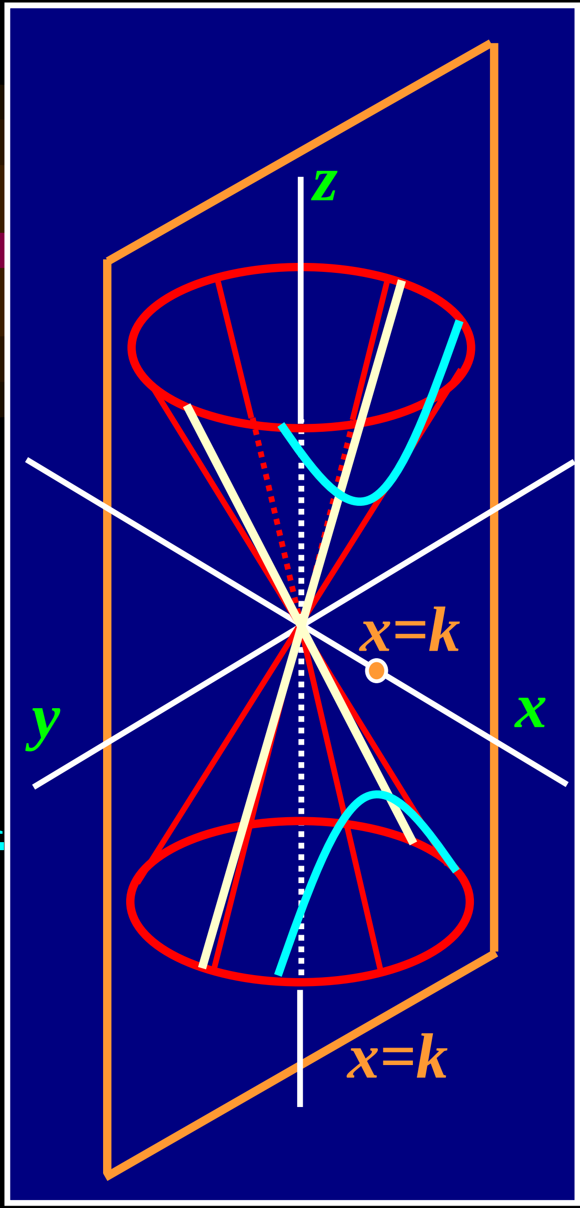
$z^2 = m^2(k^2 + y^2)$, or $y^2 = z^2 / m^2 - k^2$.

If $k = 0$, $z = \pm my$, or **Two Lines** .

$k \neq 0$, $\dfrac{z^2}{m^2k^2} - \dfrac{y^2}{k^2} = 1$, or a **Hyperbola**

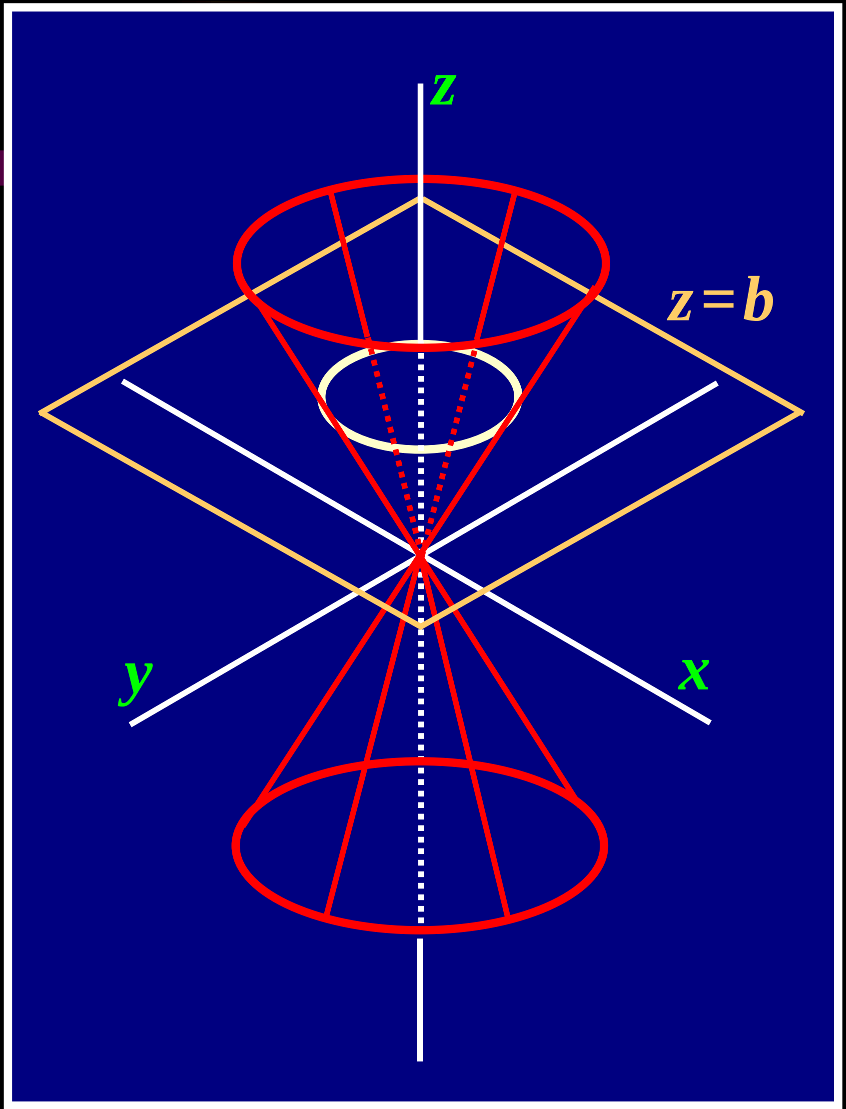
Question: What are the foci of the hyperbola?

$z^2 = m^2(x^2 + y^2)$



Conic Sections

$z^2 = m^2(x^2 + y^2)$



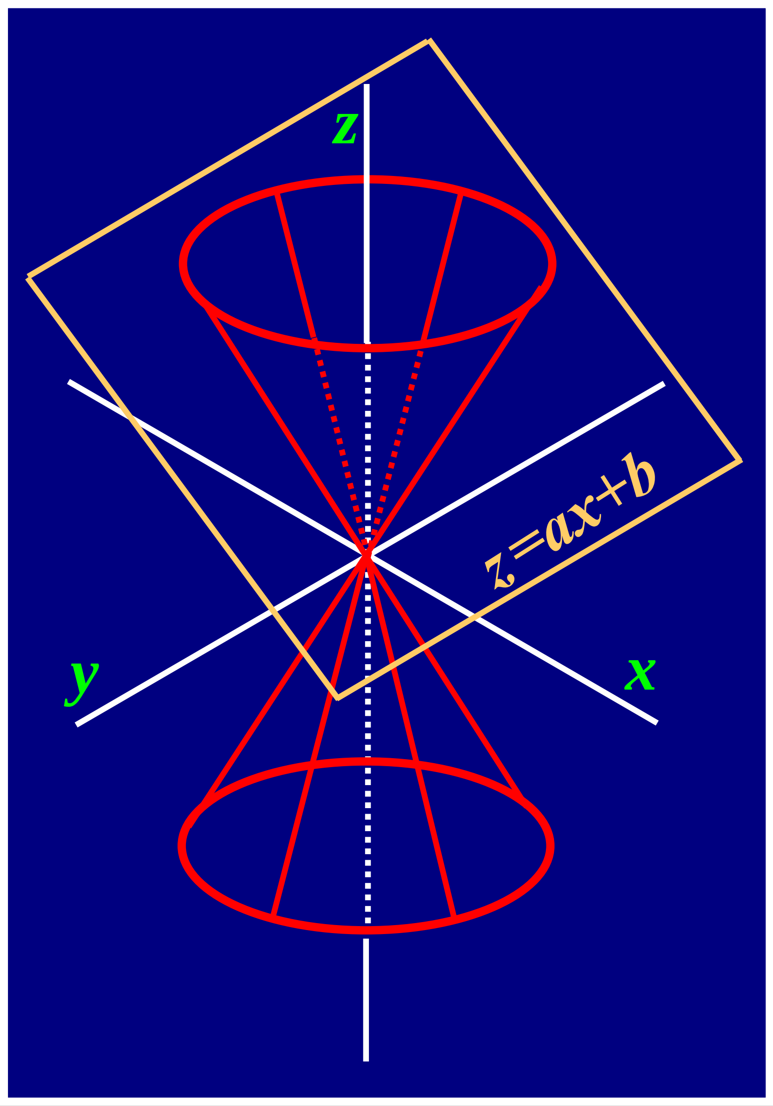
Consider the plane $z = b$. Then,

$b^2/m^2 = x^2 + y^2$, or a **circle**.

Question: What if $b = 0$?

Conic Sections

$$z^2 = m^2(x^2 + y^2)$$



Consider the plane $z = ax + b$,

$a \neq 0$. Then,

$$m^2(x^2 + y^2) = (ax + b)^2$$
$$= a^2x^2 + 2abx + b^2,$$

or,

$$m^2y^2 = (a^2 - m^2)x^2 + 2abx + b^2.$$

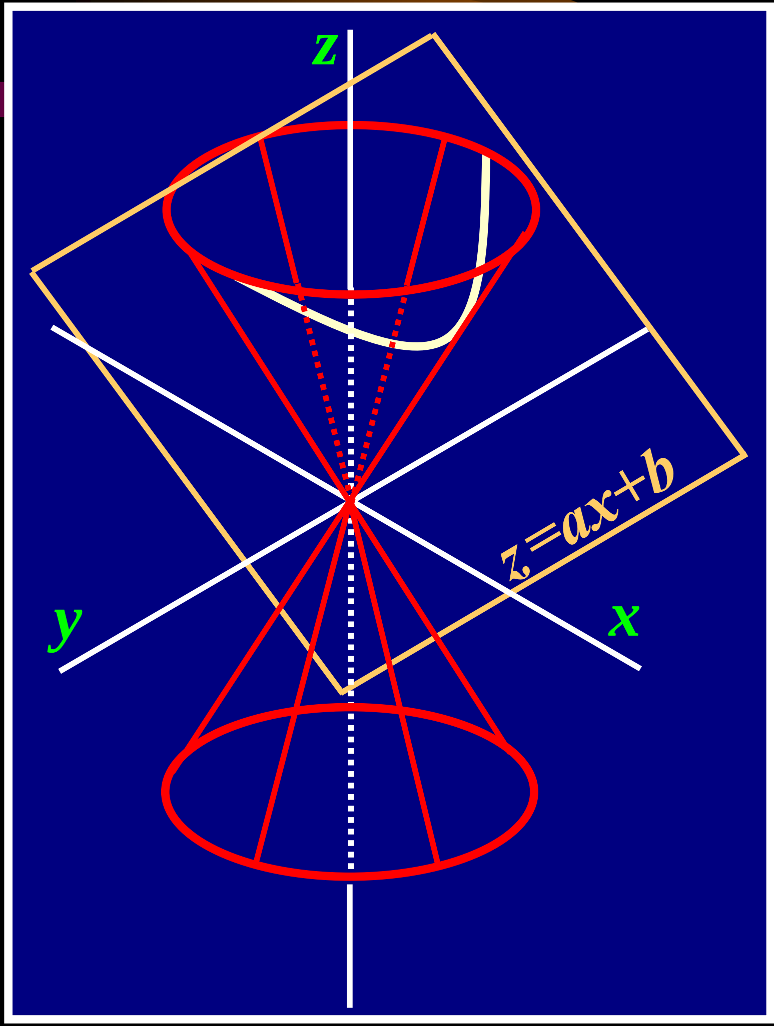
Conic Sections $m^2y^2 = (a^2 - m^2)x^2 + 2abx + b^2.$

If $a = \pm m$, then the intersection curve equals $m^2y^2 = 2abx + b^2$

or,

$$y^2 = \frac{2abx}{m^2} + \frac{b^2}{m^2},$$

a **Parabola**.



Question: What if $b = 0$?

Conic Sections

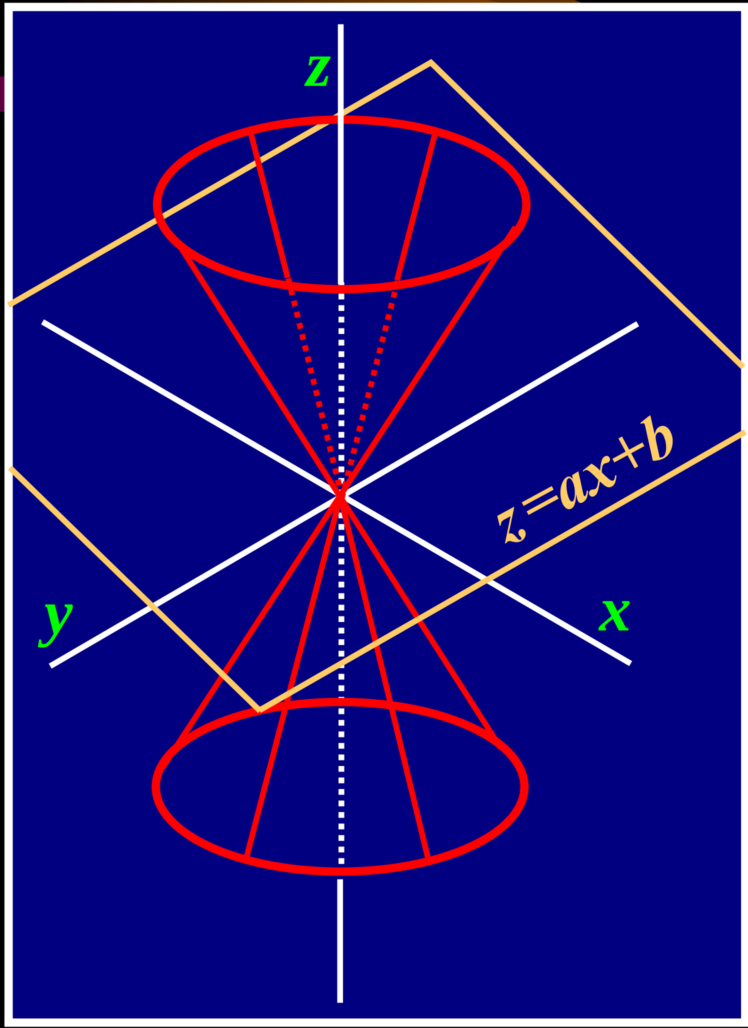
$$m^2y^2 = (a^2 - m^2)x^2 + 2abx + b^2.$$

If $a^2 < m^2$ then let $r^2 = m^2 - a^2$ and

$$m^2y^2 = -((m^2 - a^2)x^2 - 2abx) + b^2$$
$$= -(r^2x^2 - 2abx) + b^2,$$

and by completing the square,

$$m^2y^2 = -\left(rx - \frac{ab}{r}\right)^2 + \left(\frac{ab}{r}\right)^2 + b^2$$



Conic Sections

$$m^2y^2 = -\left(rx - \frac{ab}{r}\right)^2 + \left(\frac{ab}{r}\right)^2 + b^2$$

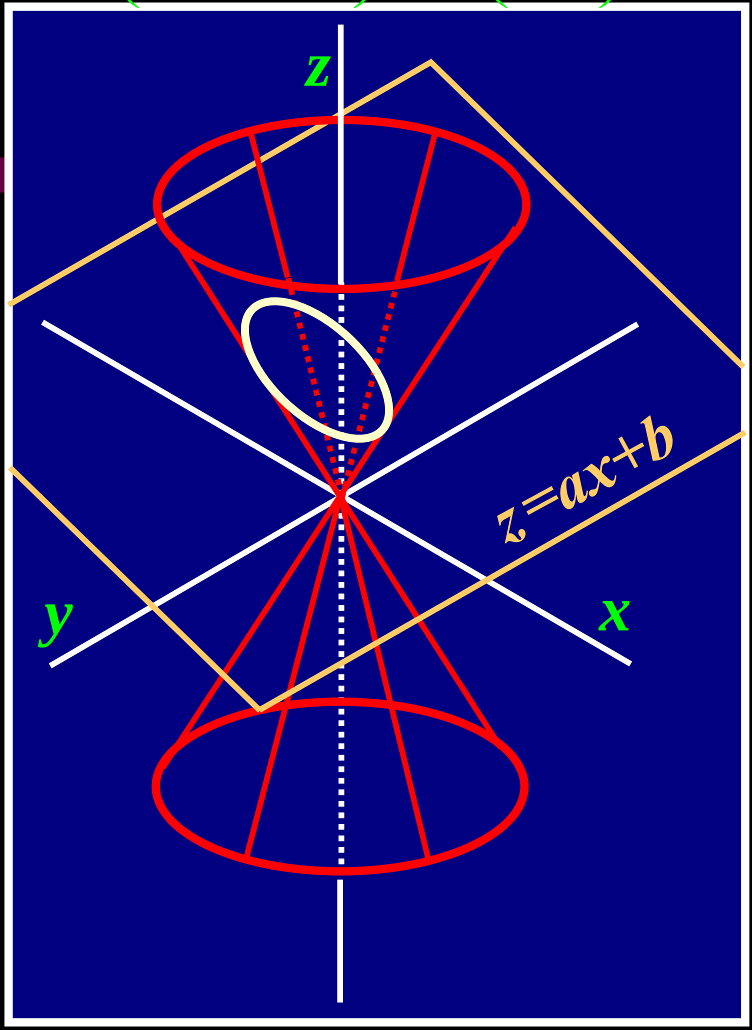
Multiplying by $r^2 = m^2 - a^2$,

$$r^2m^2y^2 + (r^2x - ab)^2 = a^2b^2 + r^2b^2$$

$$= b^2m^2,$$

and dividing by b^2m^2,

$$\frac{\left(x - \frac{ab}{m^2 - a^2}\right)^2}{\frac{b^2m^2}{\left(m^2 - a^2\right)^2}} + \frac{y^2}{\frac{b^2}{m^2 - a^2}} = 1$$



we get an **Ellipse**.

$$m^2y^2 = (a^2 - m^2)x^2 + 2abx + b^2.$$

Conic Sections

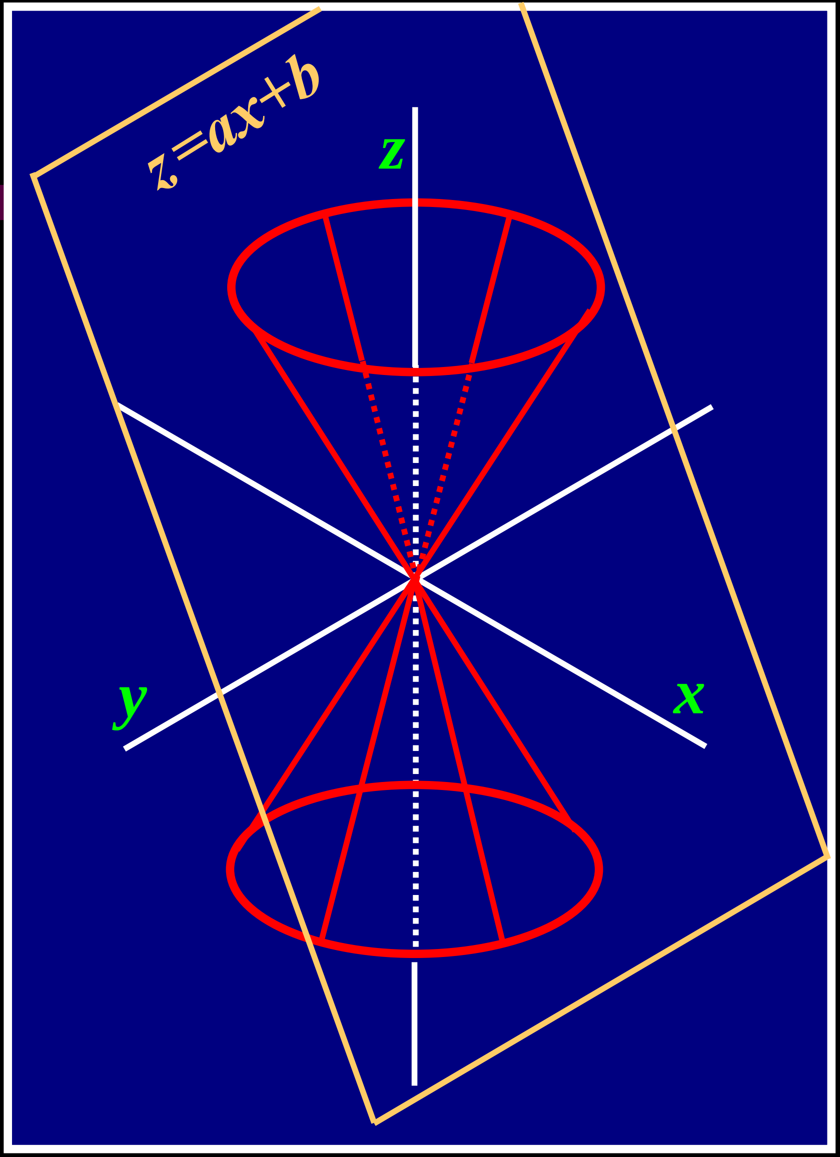
If $a^2 > m^2$ then let $r^2 = a^2 - m^2$ and

$$m^2y^2 = r^2x^2 + 2abx + b^2$$

$$= \left(rx + \frac{ab}{r}\right)^2 + b^2 - \left(\frac{ab}{r}\right)^2$$

and by multiplying by r^2,

$$\left(r^2x + ab\right)^2 - r^2m^2y^2 = b^2m^2.$$



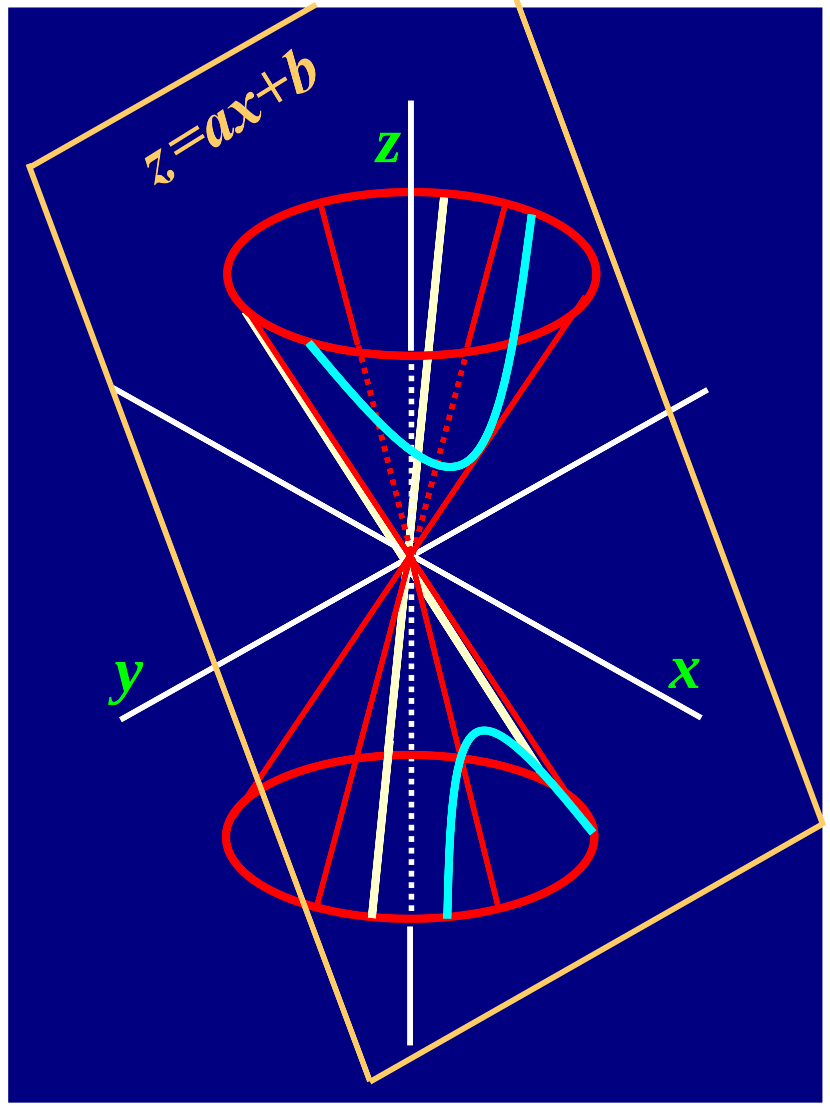
Conic Sections

$$\left(r^2x+ab\right)^2-r^2m^2y^2=b^2m^2.$$

If $b=0$, either $x=\pm my/r$ or we have **crossing lines**.

Otherwise, $b\neq 0$, divide by b^2m^2,

$$\frac{\left(x+\dfrac{ab}{a^2-m^2}\right)^2}{\dfrac{b^2m^2}{\left(a^2-m^2\right)^2}}-\frac{y^2}{\dfrac{b^2}{a^2-m^2}}=1$$



a **Hyperbola**.

Implicit Quadratic Functions as Conics

We have seen that all **conic sections** are **quadratic implicit** forms.

Question: Are all **quadratic implicit** forms **conic sections**?

Implicit Quadratic Functions as Conics

Consider the general quadratic implicit form of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Question: Is there a change of basis from (x, y) to (x', y') such that the **same graph** is drawn by the curve,

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0?$$

Definition 3.7

For the quadratic equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the quantity $B^2 - 4AC$, is called the **discriminant**.

Theorem 3.8

The **discriminant** is invariant under rotations.

Theorem 3.9

Every implicit quadratic is a conic section and

$$\text{if} \quad B^2 - 4AC \begin{cases} < 0, & \text{the curve is an ellipse,} \\ = 0, & \text{the curve is a parabola,} \\ > 0, & \text{the curve is a hyperbola.} \end{cases}$$

Proof

Since the discriminant is invariant under rotations, rotate through the special angle θ so that $B' = 0$ in the new rotated coordinate system. Then,

$$B^2 - 4AC = B'^2 - 4A'C' = -4A'C'.$$

Theorem 3.10

An implicit function $f(x, y) = 0$ is a conic section if and only if f is a second degree polynomial in x and y.

Question: How can we intuitively construct conic sections?

5 Points Construction

Question: How many degrees of freedom does the quadratic equation of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ have?

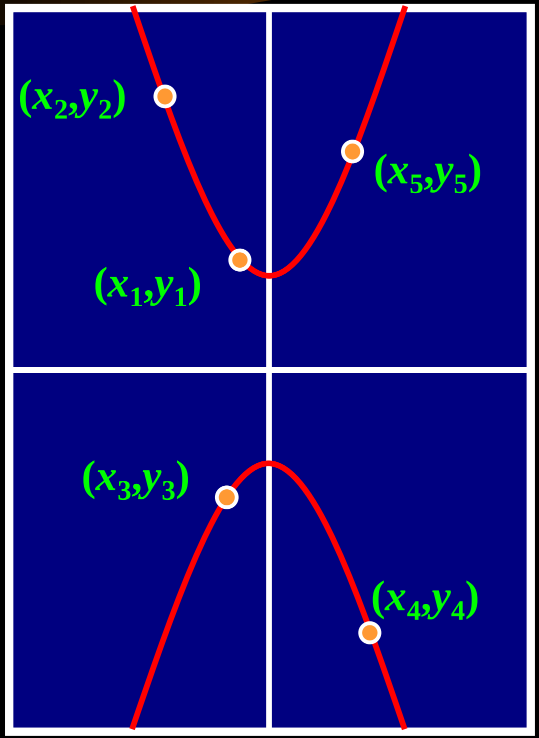
These degrees of freedom can be prescribed using five points (x_i, y_i):

$$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0$$
$$Ax_2^2 + Bx_2y_2 + Cy_2^2 + Dx_2 + Ey_2 + F = 0$$
$$Ax_3^2 + Bx_3y_3 + Cy_3^2 + Dx_3 + Ey_3 + F = 0$$
$$Ax_4^2 + Bx_4y_4 + Cy_4^2 + Dx_4 + Ey_4 + F = 0$$
$$Ax_5^2 + Bx_5y_5 + Cy_5^2 + Dx_5 + Ey_5 + F = 0$$

5 Points Construction (Cont.)

Or in **matrix** form,

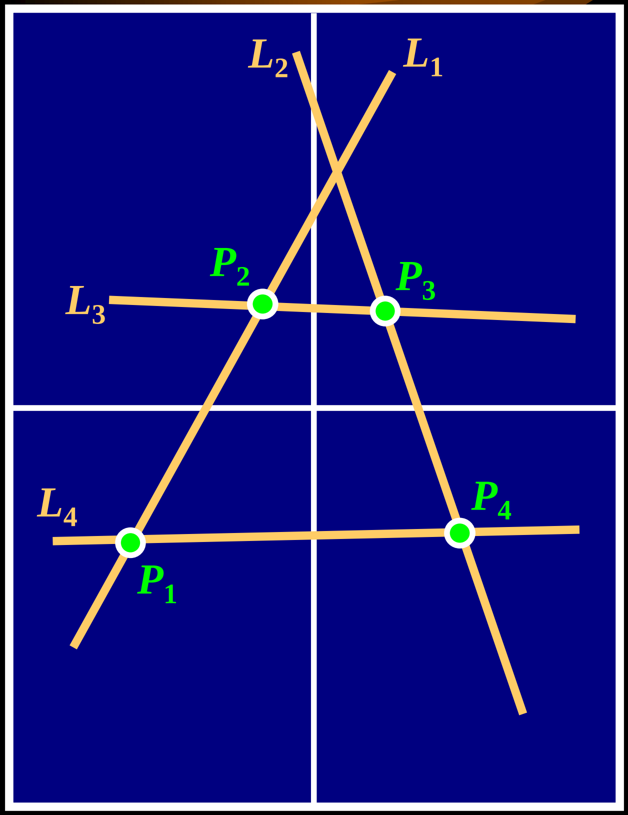
$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Questions: What is missing here?

5 Points Construction (Cont.)

Seeking a more intuitive approach, consider the four lines through the four given points,



L_1 through P_1 and P_2

L_2 through P_3 and P_4

L_3 through P_2 and P_3

L_4 through P_4 and P_1

5 Points Construction (Cont.)

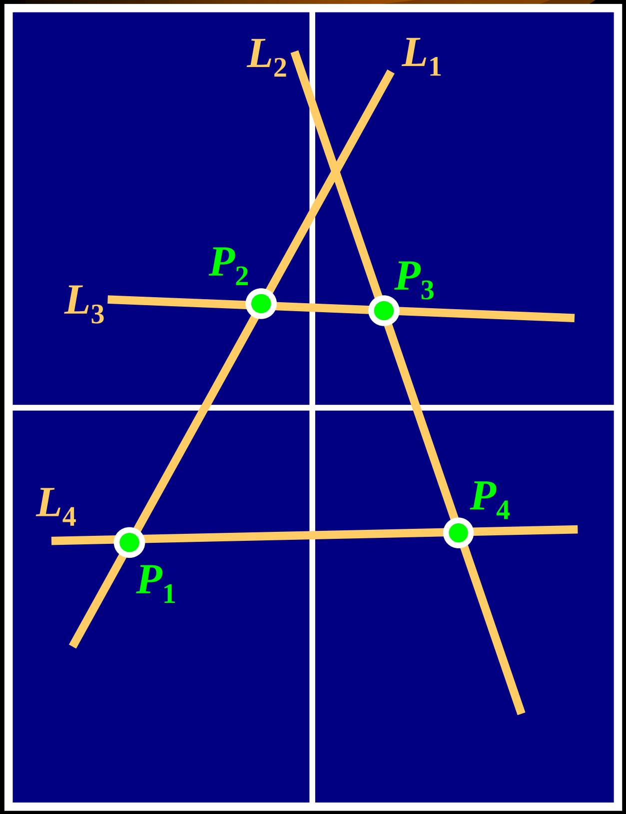
Let $L_i(x, y) = a_i x + b_i y + c_i$. Then,

$$L_1(P_j) = 0 \qquad \text{for } j = 1,2$$

$$L_2(P_j) = 0 \qquad \text{for } j = 3,4$$

$$L_3(P_j) = 0 \qquad \text{for } j = 2,3$$

$$L_4(P_j) = 0 \qquad \text{for } j = 4,1$$



5 Points Construction (Cont.)

Let $L_i(x, y)\, L_j(x, y) = (a_i x + b_i y + c_i)\,(a_j x + b_j y + c_j)$.

Observation: $L_i(x, y)\, L_j(x, y)$ is a quadratic equation in x and y.

Observation: $L_1 L_2(P_j)$, $j = 1,2,3,4$ equal zero!

Now consider the surface (c constant),

$$z = f(x, y) = L_1 L_2(x, y) + c L_3 L_4(x, y).$$

Question: To what is $f(P_j)$, $j = 1,2,3,4$ equal?

$$z = f(x,y) = L_1 L_2(x,y) + c L_3 L_4(x,y).$$

5 Points Construction (Cont.)

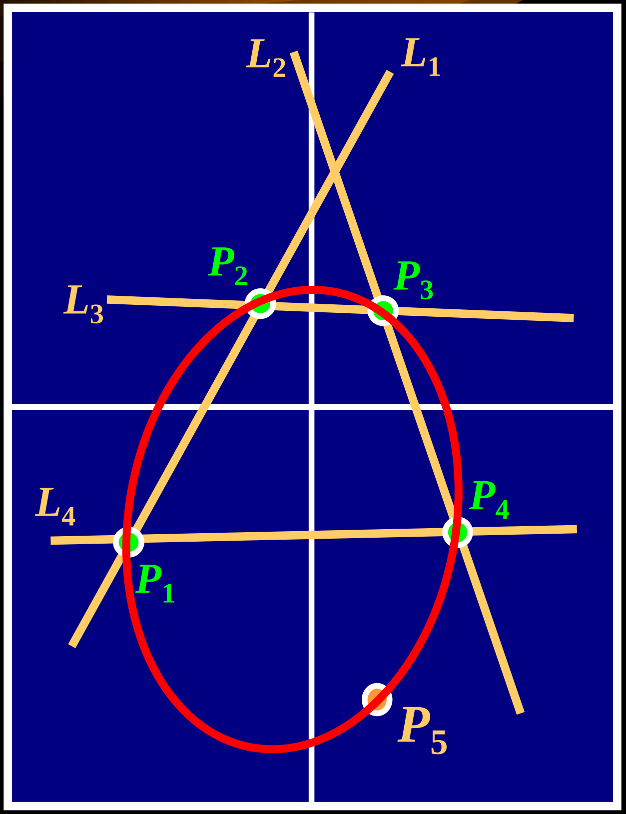
Question: How can we prescribe c?

Define a fifth point, P_5, and ensure that

$$f(P_5) = 0$$
$$= L_1 L_2(P_5) + c L_3 L_4(P_5).$$

or,

$$c = -\frac{L_1(P_5) L_2(P_5)}{L_3(P_5) L_4(P_5)}.$$



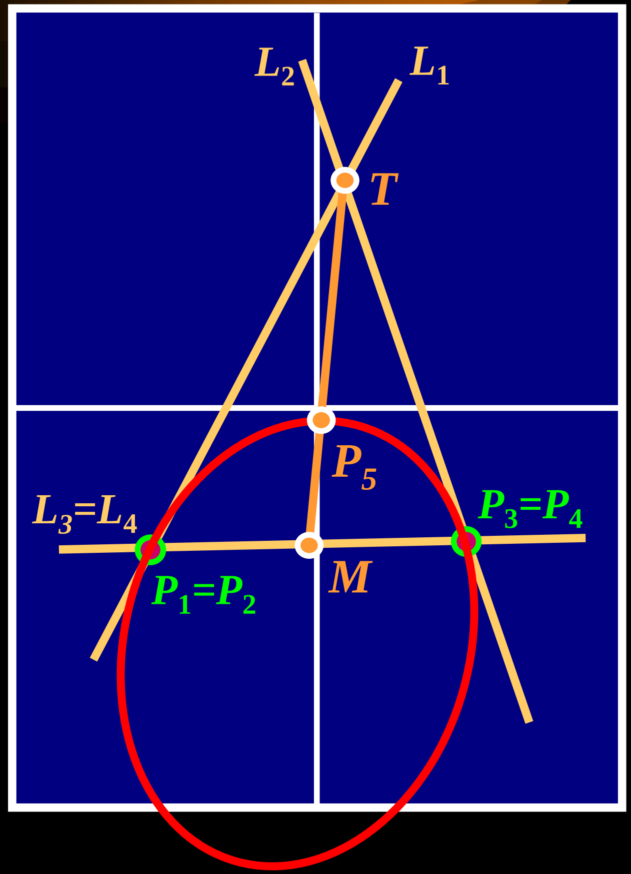
Using Tangents

Let $L_3 \equiv L_4$ and hence

$$f(x, y) = L_1 L_2 + c L_3^2$$

Question: What does the shape of $f(x, y)$ look like?

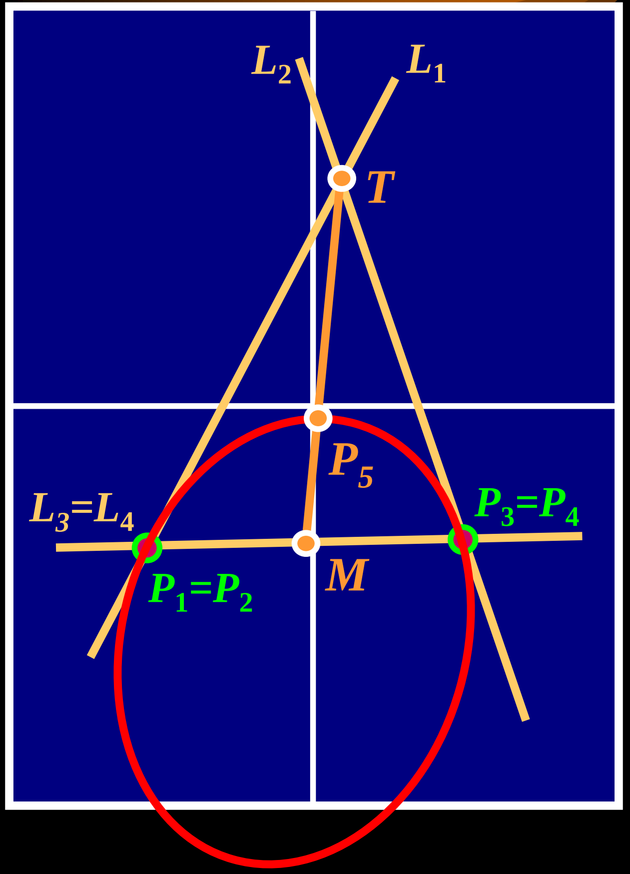
P_5 is denoted the shoulder point: $M = (P_1 + P_4) / 2$, and the ρ -conic equals $\rho = \| P_5 - M \| / \| T - M \|$.



Using Tangents

$P_5 = (1 - \rho)M + \rho T$. Then

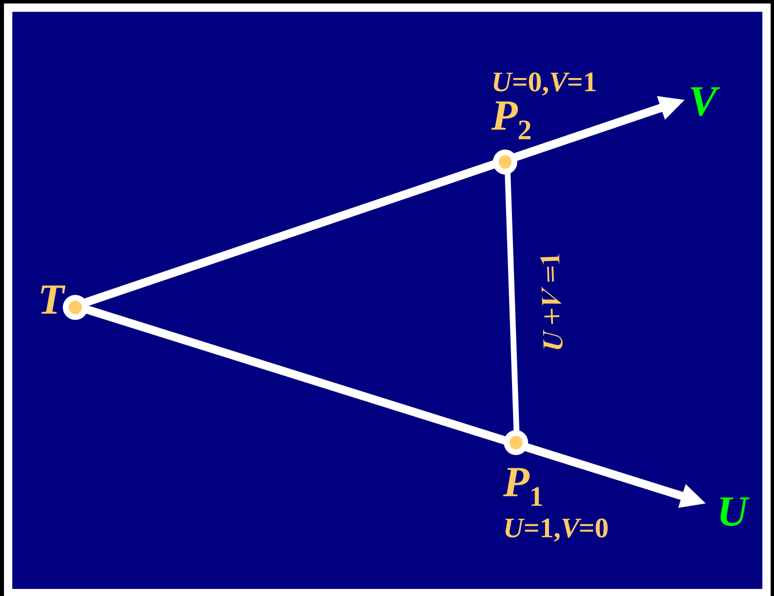
the conic is a $\begin{cases} \text{Parabola}, & \text{if } \rho = \frac{1}{2}, \\ \text{Hyperbola}, & \text{if } \rho > \frac{1}{2}, \\ \text{Ellipse}, & \text{if } \rho < \frac{1}{2}. \end{cases}$



Conic Arcs as Rational Functions

Assume T, P_1, and P_2 are not on the same line. Then, $\{\mathbf{U} = P_1 - T, \mathbf{V} = P_2 - T\}$ spans the XY plane. Every point in the XY plane can be written as

$$(u, v) = T + u(P_1 - T) + v(P_2 - T).$$

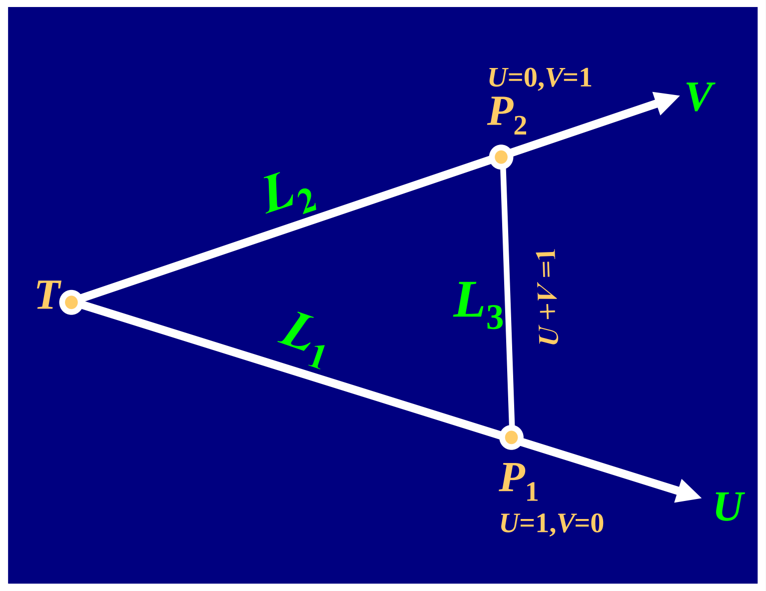


Question: Is this coordinate system rigid-motion invariant?

$$x_0 = T_x + u(P_{1,x} - T_x) + v(P_{2,x} - T_x),$$
$$y_0 = T_y + u(P_{1,y} - T_y) + v(P_{2,y} - T_y).$$

Conic Arcs as Rational Functions

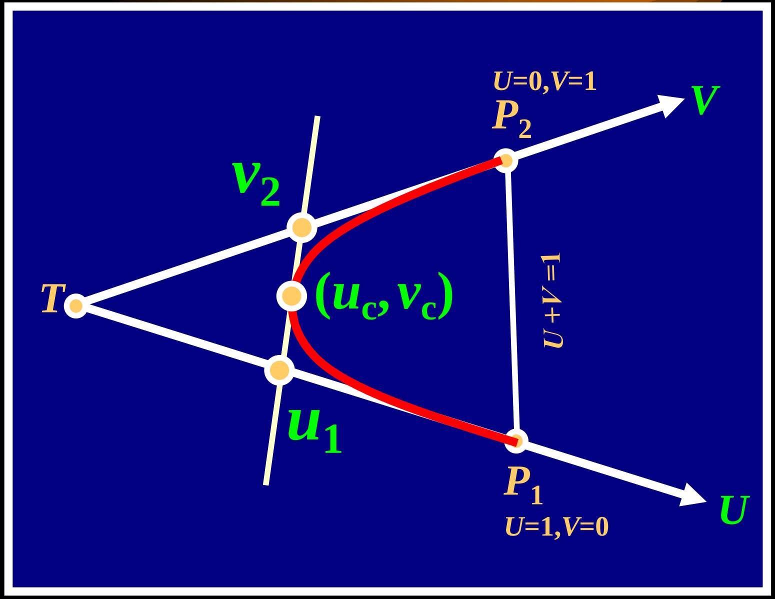
Question: What are the (u, v) coordinates of the $\overline{TP_1}$ line? The $\overline{TP_2}$ line? The $\overline{P_1P_2}$ line?



$$\mathbf{L_1}: \; v = 0, \quad \mathbf{L_2}: \; u = 0,$$
$$\mathbf{L_3}: \; u + v - 1 = 0.$$

$$\mathbf{L_1}: T + u(P_1 - T) = (1-u)T + uP_1,$$
$$\mathbf{L_2}: T + v(P_2 - T) = (1-v)T + vP_2,$$
$$\mathbf{L_3}: T + u(P_1 - T) + (1-u)(P_2 - T) = uP_1 + (1-u)P_2.$$

Conic Arcs as Rational Functions



In uv coordinates we have

$$0 = L_1 L_2 + c L_3^2$$
$$= uv + c(u + v - 1)^2.$$

Setting $c = -\lambda/(1 - \lambda)$, one gets

$$C(u,v) = (1-\lambda)uv - \lambda(u+v-1)^2 = 0.$$

For $0 \leq \lambda \leq 1$, and $0 \leq u, v$ such that $u + v \leq 1$,
$C(u, v) = 0$ is **inside** the triangle $P_1 T P_2$.

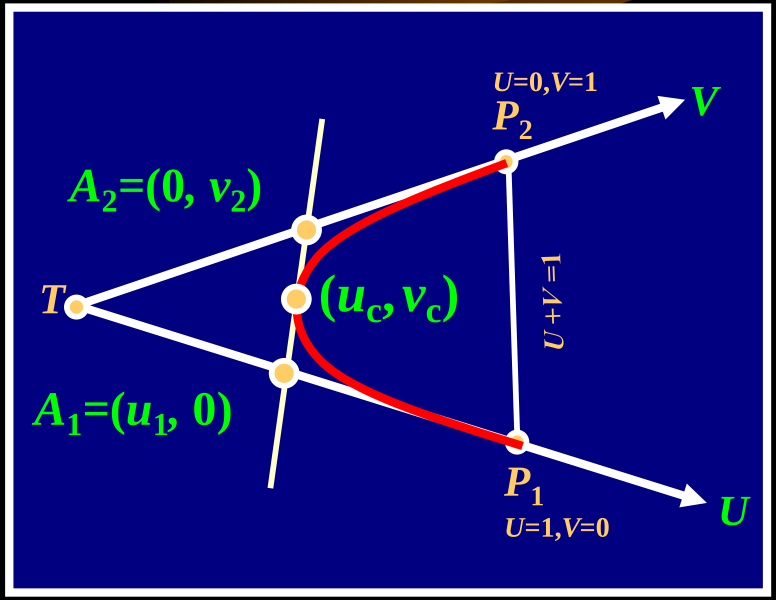
Consider a point P_c on $C(u,v) = 0$, $P_c = (u_c, v_c)$.

$$A_1 = \left(\frac{-2\lambda(u_c + v_c - 1)}{(1-\lambda)v_c - 2\lambda(u_c + v_c - 1)}, 0 \right)$$

Conic Arcs as Rational Functions

Consider the ratio

$$r_1 = \frac{\|TA_1\|}{\|P_1A_1\|} = \frac{u_1}{1-u_1} = \frac{-2\lambda(u_c + v_c - 1)}{(1-\lambda)v_c}$$



And the ratio

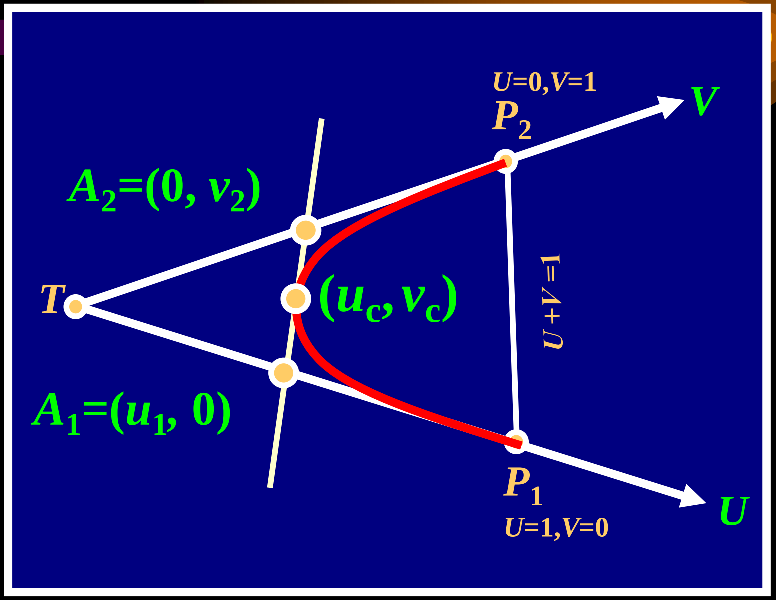
$$r_2 = \frac{\|TA_2\|}{\|P_2A_2\|} = \frac{v_2}{1-v_2} = \frac{-2\lambda(u_c + v_c - 1)}{(1-\lambda)u_c}$$

$$r_1 = \frac{-2\lambda(u_c + v_c - 1)}{(1-\lambda)v_c}, \quad r_2 = \frac{-2\lambda(u_c + v_c - 1)}{(1-\lambda)u_c}$$

Conic Arcs as Rational Functions

And consider the product of these two ratios

$$r_1 r_2 = \frac{4\lambda^2 (u_c + v_c - 1)^2}{(1-\lambda)^2 u_c v_c}$$

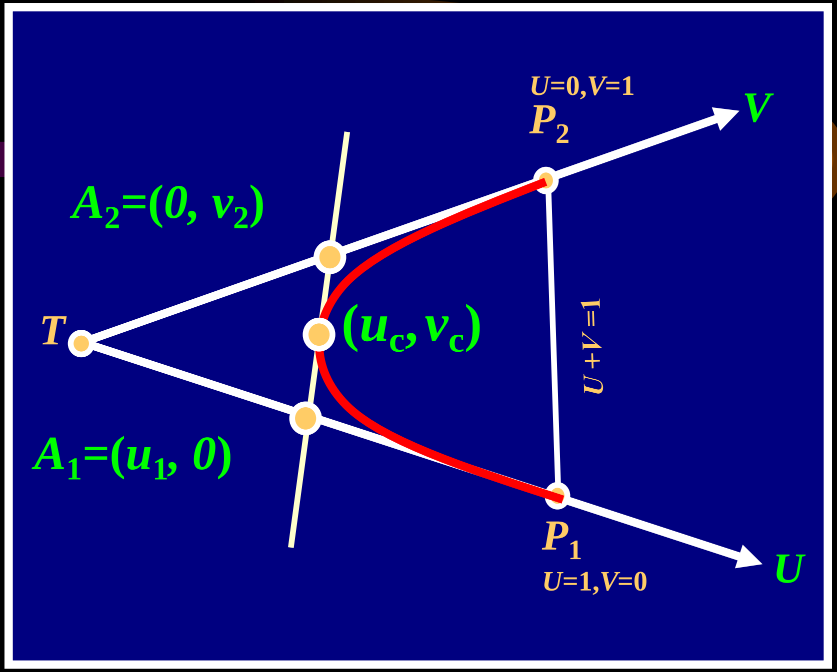


On a point (u_c, v_c), in curve $C(u, v)$, $(1-\lambda)uv = \lambda(u+v-1)^2$

or $\frac{1-\lambda}{\lambda} = \frac{(u_c + v_c - 1)^2}{u_c v_c}$ and $r_1 r_2 = \frac{4\lambda^2}{(1-\lambda)^2} \frac{1-\lambda}{\lambda} = \frac{4\lambda}{1-\lambda}$.

Theorem 3.13

If P_1, T, P_2, A_1 and A_2 are as above, then the product of the ratios



$$r_1 r_2 = \frac{\|TA_1\| \|TA_2\|}{\|A_1P_1\| \|A_2P_2\|} = \frac{uv}{(1-u)(1-v)}$$

is a constant for the whole conic section.

$$u_c = \frac{r_1}{r_1 + r_2 + r_1 r_2 \frac{1-\lambda}{2\lambda}}$$

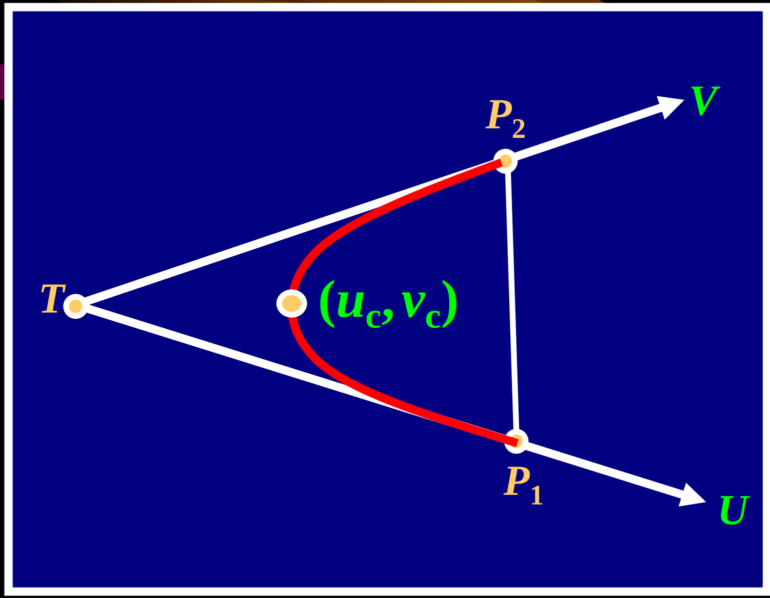
Conic Arcs as Rational Functions

Because $r_1 r_2 = \dfrac{4\lambda}{1-\lambda}$, we have,

$$u_c = \frac{r_1}{r_1 + r_2 + 2}$$ and similarly $$v_c = \frac{r_2}{r_1 + r_2 + 2}.$$

Conic Arcs as Rational Functions

Going back to the conic curve, we have



$$\gamma = T + u_c(P_1 - T) + v_c(P_2 - T)$$

$$= T + \frac{r_1}{r_1 + r_2 + 2}(P_1 - T) + \frac{r_2}{r_1 + r_2 + 2}(P_2 - T)$$

$$= \frac{r_1 P_1 + 2T + r_2 P_2}{r_1 + 2 + r_2}.$$

$$\gamma = \frac{r_1 P_1 + 2T + r_2 P_2}{r_1 + 2 + r_2}.$$

Conic Arcs as Rational Functions

In order to parameterize γ as a rational form $\gamma(t)$, $t \in (a, b)$, r_1 and r_2 must satisfy the following constraints,

1. $r_1 r_2 = k$, $k = 4\lambda/(1-\lambda)$ constant.
2. $r_1(t)$, $r_2(t)$ map (a, b) to $(0, \infty)$ and $(\infty, 0)$.
3. $r_1(t)$, $r_2(t)$ must be monotone for $t \in (a, b)$.

Question: Why $r_1(t)$, $r_2(t)$ map to $(0, \infty)$? Why is there a monotonicity constraint?

Conic Arcs as Rational Functions

One possible solution for $r_1(t)$, $r_2(t)$ is $r_1(t) = \dfrac{w_1(b-t)}{w(t-a)}$ and $r_2(t) = \dfrac{w_2(t-a)}{w(b-t)}$:

$$K = \frac{4\lambda}{1-\lambda} = r_1(t) r_2(t)$$

$$= \frac{w_1(b-t)}{w(t-a)} \frac{w_2(t-a)}{w(b-t)}$$

$$= \frac{w_1 w_2}{w^2}.$$

or 1 is verified. 2 and 3 are trivial to verify as well.

Conic Arcs as Rational Functions

Then,

$$\gamma(t) = \frac{r_1 P_1 + 2T + r_2 P_2}{r_1 + 2 + r_2} = \frac{\frac{w_1(b-t)}{w(t-a)} P_1 + 2T + \frac{w_2(t-a)}{w(b-t)} P_2}{\frac{w_1(b-t)}{w(t-a)} + 2 + \frac{w_2(t-a)}{w(b-t)}}$$

$$= \frac{w_1(b-t)^2 P_1 + 2w(t-a)(b-t)T + w_2(t-a)^2 P_2}{w_1(b-t)^2 + 2w(t-a)(b-t) + w_2(t-a)^2}.$$

Hence, every conic section can be written as a rational quadratic parametric function.

Example 3.19 (Arc of a circle)

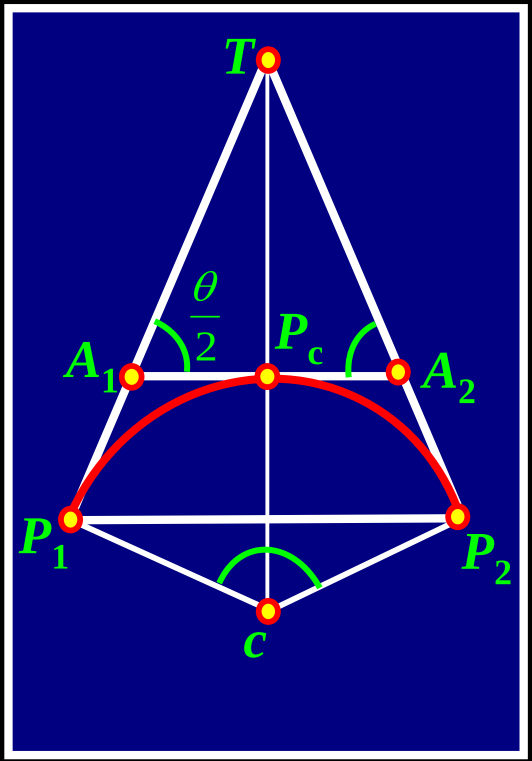
Assume a circle of radius r spanning α degs. For $r_i = 1,2$,

$$r_i = \frac{\|TA_i\|}{\|A_iP_i\|} = \frac{\|TA_i\|}{\|A_iP_c\|} = \frac{\|TA_i\|}{\|TA_i\|\cos(\theta/2)} = \frac{1}{\cos(\theta/2)}$$

Thus,

$$K = \frac{w_1 w_2}{w^2} = r_1 r_2 = \frac{1}{\cos^2(\theta/2)}$$

Question: What will be the effect, if any, of $w_1 \leftarrow w_1\,\alpha$, $w_2 \leftarrow w_2/\alpha$?



Homogeneous Coordinates

The rational form of quadratics equals,

$$\gamma(t) = \frac{w_1(b-t)^2 P_1 + 2w(t-a)(b-t)T + w_2(t-a)^2 P_2}{w_1(b-t)^2 + 2w(t-a)(b-t) + w_2(t-a)^2}.$$

Let $\theta_1(t) = (b - t)^2$, $\theta_2(t) = 2(t - a)(b - t)$, $\theta_3(t) = (t - a)^2$. Then,

$$\gamma(t) = \frac{w_1 P_1 \theta_1(t) + wT\theta(t) + w_2 P_2 \theta_2(t)}{w_1\theta_1(t) + w\theta(t) + w_2\theta_2(t)}$$

$$\equiv (w_1 P_1, w_1)\theta_1(t) + (wT, w)\theta(t) + (w_2 P_2, w_2)\theta_2(t).$$