Computer Aided Geometric Design

Class Exercise #3

- Differential Geometry of Curves
Q.1

Let \( c: I \rightarrow \mathbb{R}^3 \) be an arc-length parameterized curve. Prove:

1. \( \kappa(s) \equiv 0 \iff c(s) \) is a straight line.
2. \( \tau(s) \equiv 0 \iff c(s) \) is a plane curve.
Solution - Part 1 ($\kappa \equiv 0$)

Recall that $\kappa$ is the magnitude of the curvature vector:

$$c''(s) = \kappa(s)N(s).$$

Therefore,

$$c''(s) \equiv (0,0,0),$$

and by integration:

$$c'(s) \equiv (a_1, a_2, a_3),$$

for some real constants $a_1, a_2, a_3$. 

Solution - Part 1 ($\kappa \equiv 0$)

This already means: The direction of the curve is in fact a constant direction.

To verify that this is a line, we integrate again and:

\[
c(s) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3)
= (b_1, b_2, b_3) + s(a_1, a_2, a_3).
\]
Solution - Part 1 ($\kappa \equiv 0$)

and we obtained the parameterization of the straight line passing through $(b_1, b_2, b_3)$ with the direction $(a_1, a_2, a_3)$:

$(b_1, b_2, b_3)$

$(a_1, a_2, a_3)$

(Conversely – the same with differentiation!)
Solution - Part 1 ($\kappa \equiv 0$)

Question:

Where did we use the fact that the curve was \textit{arc-length} parameterized?

Is a \textit{general} parameterization $c(t)$ with $\kappa(t) \equiv 0$ not necessarily a straight line?
Solution - Part 1 ($\kappa \equiv 0$)

Answer:

This should also be true for non arc-length parameterization...

Recall that for non arc-length:

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$
Solution - Part 1 ($\kappa \equiv 0$)

if $\kappa \equiv 0$ then:

$$c'(t) \times c''(t)$$

is the zero vector.

When does a cross product vanish?
Solution - Part 1 ($\kappa \equiv 0$)

when $c'$ and $c''$ are the same (or opposite) directions.

We understand this as follows:
The tangent never changes direction, only magnitude.
Solution - Part 2 ($\tau \equiv 0$)

How can we show that a curve is planar?

(without the knowledge of this being equivalent to $\tau \equiv 0$ which we are trying to prove....)
Solution - Part 2 ($\tau \equiv 0$)

We must show that the curve is contained in some plane.

A general (implicit) representation of a plane:

$$\{ p \in \mathbb{R}^3 : \langle \nu, p \rangle = \alpha \}$$

where $\nu \in \mathbb{R}^3$ is a constant (non-zero) vector, and $\alpha \in \mathbb{R}$. 
Solution - Part 2 ($\tau \equiv 0$)

In our case: we show that:

$$\langle B_0, c(s) \rangle = \text{const}$$

where $B_0 \in \mathbb{R}^3$ is the bi-normal at some point.
Solution - Part 2 ($\tau \equiv 0$)

let $B_0 = B(s_0)$ for some $s_0 \in I$.

since:

$$\tau \equiv 0,$$

we have:

$$B'(s) = -\tau(s)N(s) = 0$$

which means $B(s)$ is a constant vector.
Solution - Part 2 \((\tau \equiv 0)\)

Now we look at:

\[
\frac{d}{ds} \langle c(s), B_0 \rangle = \langle c'(s), B_0 \rangle = 0,
\]

and therefore:

\[
\langle c(s), B_0 \rangle = \text{const}
\]

and \(c(t)\) is planar.
Conversely:

If a curve is planar then it is in fact contained in the (constant) osculating plane.

This means that the bi-normal is constant:

\[ 0 = B'(s) = -\tau N(s) \Rightarrow \tau(s) = 0. \]
Solution - Part 2 ($\tau \equiv 0$)

Points to think about:

- Is this the case for non arc-length parameterizations?
- What if $N(s) = 0$? Is the torsion well defined?
Q.2

Let \( c: I \to \mathbb{R}^3 \) be an arc-length parameterized curve. Prove that if all normal vectors to the curve pass through a point, then \( c(s) \) is an arc of a circle.
Solution

It suffices to show that:

- \( c(s) \) is planar.
- \( c(s) \) has constant curvature.
Solution

The assumption on the normal vectors means there exists a scalar function $\lambda(s)$ such that:

$$c(s) + \lambda(s)N(s) = \rho$$

Where $\rho$ is a constant point in $\mathbb{R}^3$. 
Solution

Differentiating gives:

\[ c'(s) + \lambda'(s)N(s) + \lambda(s)N'(s) = 0. \]

Which means:

\[ T(s) + \lambda'(s)N(s) - \lambda(s)\kappa(s)T(s) + \lambda(s)\tau(s)B(s) = 0 \]
Solution

or:

\[
(1 - \lambda(s)\kappa(s))T(s) + \lambda'(s)N(s) \\
+ \lambda(s)\tau(s)B(s) = 0
\]
Solution

We expressed the zero vector as a linear combination of $T, N, B$ at every $s$.

Therefore:

\[ 1 - \lambda(s)\kappa(s) = 0 \Rightarrow \lambda(s) = \frac{1}{\kappa(s)}, \]
\[ \lambda'(s) = 0 \Rightarrow \lambda = \text{const} \Rightarrow \kappa(s) = \text{const}. \]
\[ \lambda(s)\tau(s) = 0 \Rightarrow \lambda(s) = 0 \text{ or } \tau(s) = 0. \]
Solution

In the case $\lambda(s) = 0$, we get $\lambda(s) \equiv 0$, and the entire curve is a point.

In the other case: $\tau(s) \equiv 0$, as required.
Let \( c(t) \) be a parameterized quadratic curve. Prove that \( c(t) \) is planar.
Solution

Again, it suffices to show that the bi-normal is constant.

This shall imply that the osculating plane is constant, and the curve never leaves it.
Solution

c(t) is quadratic. We may assume:

c(t) = (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2, c_0 + c_1 t + c_2 t^2)
Solution

Differentiating gives:

\[ c'(t) = (a_1 + 2a_2 t, b_1 + 2b_2 t, c_1 + 2c_2 t), \]

\[ c''(t) = (2a_2, 2b_2, 2c_2). \]
Solution

We know that $c'(t)$ has the same direction as the unit tangent $T$.

Does $c''(t)$ has the same direction as the unit normal $N$?
Solution

Not necessarily. It is true for arc-length parameterizations.

(in fact for any constant speed...)

But:

$c''(t)$ is contained in the plane spanned by $T, N$. 
Solution

This can be shown analytically.

Geometrically, it is explained by the fact that \( c''(t) \) changes the tangent in direction and in magnitude.

Therefore: the direction \( c' \times c'' \) is in the bi-normal direction!
Solution

It suffices to check that \( c' \times c'' \) is constant. For example, the first component of this cross product is:

\[
(b_1 + 2b_2 t)2c_2 - (c_1 + 2c_2 t)2b_2 = 2b_1 c_2 - 2b_2 c_1
\]

which does not depend on \( t \). Others are shown similarly.
Solution

Thus, the bi-normal direction is constant – and the curve is planar.
Q.4

Consider the curves:

\[ c(t) = (t \cos(t), t \sin(t)), \]
\[ t \in (0, 4\pi). \]

and:

\[ d(t) = (8 \cos(t), 8 \sin(t)), \]
\[ t \in (-\frac{\pi}{2}, 0). \]
Q.4

Find \textit{rigid motions} on \( d(t) \) such that it will be pieced together with \( c(t) \) and the resulting curve is:

1. Continuous (\( C^0 \)).
2. Geometrically \( C^1 \).
3. \( C^1 \)
Solution

The curves are the following:
Solution

c(t) has a tangent:

c'(t) = (cos(t) − tsin(t), sin(t) + tcos(t)).

and at \( t = 4\pi \):

\[
c(4\pi) = (4\pi, 0),
\]

\[
c'(4\pi) = (1, 4\pi).
\]
Solution

d(t) is the lower right quarter of the unit circle, inflated to a radius of length 8.

For continuity we simply translate \( d(t) \) by:

\[
p = (4\pi, 0) - (0, -8) = (4\pi, 8)
\]
Solution

This gives:
Solution

For Geometrically $C^1$ we need first to rotate $d(t)$ such that the tangents have the same direction.

Since:

$$c'(4\pi) = (1, 4\pi).$$

and:

$$d'(\frac{-\pi}{2}) = (8, 0).$$
Solution

The angle between these tangents is given by:

\[ \langle (1,4\pi), (8,0) \rangle = 8 \cdot \sqrt{1 + (4\pi)^2} \cos(\theta), \]

\[ \Rightarrow \cos(\theta) = \frac{1}{\sqrt{1 + (4\pi)^2}} \]
Solution

Which gives:

\[ \theta \approx 1.4914 \text{rad} \]

or:

\[ \theta \approx 85.4501^\circ. \]

It remains to rotate and translate:

\[ \tilde{d}(t) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} d(t) + q, \]
**Solution**

We have the rotation angle, but what is $q$, the translation vector?

After the rotation we have:
Solution

The image of \((0, -8)\) under the rotation is:

\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
0 \\
-8
\end{pmatrix}
\approx
\begin{pmatrix}
8.224 \\
2.4971
\end{pmatrix}
\]

Then:

\[q = (4\pi, 0) - (8.224, 2.4971)\]
Solution

Finally:
Solution

In order for the resulting curve to be $C^1$, we need to change the magnitude of the tangent.

Denote by $r$ the speed ratio:

$$r = \frac{\tilde{d}'(-\frac{\pi}{2})}{c'(4\pi)}$$
Solution

Re-parameterize $\tilde{\mathcal{d}}$ as follows:

Instead of:

$$\tilde{\mathcal{d}}(t), t \in [a, b],$$

take:

$$\tilde{\mathcal{d}}(rt), t \in \left[\frac{a}{r}, \frac{b}{r}\right].$$

This results in the same trace, but the speed is factored by $r$, as required.
Solution

Re-parameterize $\tilde{d}$ as follows:
Instead of:
\[ \tilde{d}(t), \ t \in [a, b], \]
take:
\[ \tilde{d}(rt), \ t \in \left[ \frac{a}{r}, \frac{b}{r} \right]. \]

This results in the same trace, but the speed is factored by $r$, as required.
Solution

Some points to think about:

- How do we re-parameterize as a *single* curve?
- In practice – how do we rotate without explicitly invoking \( \text{arccos} \)?