Computer Aided Geometric Design

Class Exercise #5

- Bezier Curves
Q.1

Let $f: [0,1] \to \mathbb{R}$, given as a linear combination of Bernstein polynomials of degree $n$.

1. Is it a parametric Bezier curve, in the regular sense?

2. Find control points in $\mathbb{R}^2$ such that the graph of $f$ is the corresponding Bezier curve.
Solution – Part 1

Since \( f \) is a linear combination of Bernstein polynomials, it is of the form:

\[
f(t) = \sum_{i=0}^{n} f_i \theta_{i,n}(t),
\]
Solution – Part 1

where:

- $f_0, \ldots, f_n$ are real numbers.
- $\theta_{i,n}(t) = \binom{n}{i}(1 - t)^{n-i}t^i$

The image is scalar $\Rightarrow$ In the “parametric” sense it is only a degenerate “curve” on the real line.
Solution – Part 1

All we know:
It is a segment contained in the segment \([f_{\text{min}}, f_{\text{max}}]\).

Question:
Under what conditions is the image equal to the segment \([f_{\text{min}}, f_{\text{max}}]\)?
Solution – Part 2

We are now interested in the graph of $f$: a curve in explicit representation.

This means that the parametric curve is now:

$\gamma: [0,1] \rightarrow \mathbb{R}^2$

$\gamma(t) = (x(t), y(t)) = (t, f(t))$. 
Solution – Part 2

For this to be a Bezier curve, we seek control points:

\[ \{P_i\}_{i=0,...,n} \subset \mathbb{R}^2, \]

such that for all \( t \in [0,1] \):

\[ (t, f(t)) = \sum_{i=0}^{n} P_i \theta_{i,n}(t), \]
Solution – Part 2

The candidates for the $y$ coordinates are naturally the original $f_i$’s, since:

\[ \forall t \in [0,1]: \]

\[ \sum_{i=0}^{n} f_i \theta_{i,n}(t) = f(t) = \sum_{i=0}^{n} P_{i,y} \theta_{i,n}(t), \]
Solution – Part 2

For the $P_{i,x}$’s we need to demand:

\[ \forall t \in [0,1]: \]

\[ t = \sum_{i=0}^{n} P_{i,x} \theta_{i,n}(t), \]
Solution – Part 2

∀t ∈ [0,1]:

\[ t = \sum_{i=0}^{n} P_i x \theta_{i,n} (t), \]

- Can we always do that?
- How?
Solution – Part 2

Recall from lectures the Bernstein approximation for a scalar function on [0,1]:

\[ B_n(f; x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \theta_{i,n}(x) \]
Solution – Part 2

The Bernstein approximation is exact on constants and on linear function (Why?)

Therefore:

\[ x = \sum_{i=0}^{n} \frac{i}{n} \theta_{i,n}(x) \]
Solution – Part 2

Thus the control polygon we are looking for is:

\[ P_i = \left( \frac{i}{n}, f_i \right), \quad i = 0, \ldots, n \]

and it is guaranteed to produce the graph of \( f \) as its parametric Bezier curve.
Q.2

(Cubic Bernstein – Monomials conversions)

1. Find the **Bernstein form** of the polynomial:

   \[ p(x) = x^3 + 4x^2 + x - 3. \]

2. Find the **standard** form of the polynomial:

   \[ q(x) = \theta_{0,3}(x) + 2\theta_{1,3}(x) - 4\theta_{2,3}(x) + \theta_{3,3}(x) \]
Solution – Part 1

We shall give a general solution for the cubic polynomials.

Converting Standard $\rightarrow$ Bernstein:

Write the elements of the standard basis as linear combinations of the Bernstein basis elements.
Solution – Part 1

Recall:

\[
\theta_{0,3}(x) = (1 - x)^3 = 1 - 3x + 3x^2 - x^3
\]

\[
\theta_{1,3}(x) = 3x(1 - x)^2 = 3(x - 2x^2 + x^3)
\]

\[
\theta_{2,3}(x) = 3x^2(1 - x) = 3(x^2 - x^3)
\]

\[
\theta_{3,3}(x) = x^3
\]
## Solution – Part 1

<table>
<thead>
<tr>
<th></th>
<th>$1 - 3x + 3x^2 - x^3$</th>
<th>$3(x - 2x^2 + x^3)$</th>
<th>$3(x^2 - x^3)$</th>
<th>$x^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
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<tr>
<td>$x$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
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<tr>
<td>$x^2$</td>
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<td>$\frac{1}{3}$</td>
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<tr>
<td>$x^3$</td>
<td>$0$</td>
<td>$0$</td>
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<td>$1$</td>
</tr>
</tbody>
</table>
Solution – Part 1

Then our transformation matrix is:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{3} & 0 & 0 & 0 \\
1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}$$
Solution – Part 1

We apply to our polynomial:

\[ p(x) = x^3 + 4x^2 + x - 3 \]

has coordinates in the standard basis:

\[
\begin{pmatrix}
-3 \\
1 \\
4 \\
1
\end{pmatrix}
\]
Solution – Part 1

The coordinates in the Bernstein Basis are given by:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & \frac{1}{3} & 0 & 0 \\
1 & \frac{2}{3} & \frac{1}{3} & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
-3 \\
1 \\
4 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
-3 \\
-\frac{8}{3} \\
-1 \\
3
\end{pmatrix}
\]
Solution – Part 1

Which means:

\[ p(x) = x^3 + 4x^2 + x - 3 \]

\[ = -3\theta_{0,3}(x) - \frac{8}{3}\theta_{1,3}(x) - \theta_{2,3}(x) + 3\theta_{3,3}(x) \]
Solution – Part 2

Converting Bernstein → Standard:

Write the elements of the Bernstein basis as linear combinations of the standard basis elements.

(or... take the inverse of the previous matrix!)
## Solution – Part 2

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$x$</th>
<th>$x^2$</th>
<th>$x^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 3x + 3x^2 - x^3$</td>
<td>$1$</td>
<td>$-3$</td>
<td>$3$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$3(x - 2x^2 + x^3)$</td>
<td>$0$</td>
<td>$3$</td>
<td>$-6$</td>
<td>$3$</td>
</tr>
<tr>
<td>$3(x^2 - x^3)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$3$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$x^3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Solution – Part 2

Then our transformation matrix is:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 & 0 \\
3 & -6 & 3 & 0 & 0 \\
-1 & 3 & -3 & 1 & 0
\end{pmatrix}
\]
Solution – Part 2

We apply to our polynomial:

\[ q(x) = \theta_{0,3}(x) + 2\theta_{1,3}(x) - 4\theta_{2,3}(x) + \theta_{3,3}(x) \]

has coordinates in the Bernstein basis:

\[
\begin{pmatrix}
1 \\
2 \\
-4 \\
1
\end{pmatrix}
\]
Solution – Part 2

The coordinates in the standard basis are given by:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 & 0 \\
3 & -6 & 3 & 0 & 0 \\
-1 & 3 & -3 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
-4 \\
1
\end{pmatrix}
=
\begin{pmatrix}
1 \\
3 \\
-21 \\
18
\end{pmatrix}
\]
Solution – Part 2

Which means:

\[ q(x) = \theta_{0,3}(x) + 2\theta_{1,3}(x) - 4\theta_{2,3}(x) \]
\[ = 1 + 3x - 21x^2 + 18x^3. \]
Q.3

Let $\gamma(t)$ be a Bezier curve of order $n$. Give a formula for $\gamma^{(j)}(t)$, for all $j = 1, \ldots, n$. 
Solution

In lectures we saw the following:

\[ \theta'_{i,n}(t) = n \left( \theta_{i-1,n-1}(t) - \theta_{i,n-1}(t) \right) \]

Which gave:

\[ \gamma'(t) = \sum_{i=0}^{n-1} Q_i \theta_{i,n-1}(t), \]

\[ Q_i = n(P_{i+1} - P_i). \]
Solution

Now, the derivative of order $j$ must be a *polynomial curve of degree* $n - j$.

Then it must have the form:

$$\gamma^{(j)}(t) = \sum_{i=0}^{n-j} Q_{j,i} \theta_{i,n-j}(t)$$

(The derivative of a Bezier curve is also a Bezier curve!)
Solution

\[ \gamma^{(j)}(t) = \sum_{i=0}^{n-j} Q_{j,i} \theta_{i,n-j}(t) \]

But what are good candidates for the \( j^{th} \) control polygon, \( Q_{j,i}, i = 0,1, \ldots, n - j \)?
Solution

For the first derivative we had:

\[ Q_i = n(P_{i+1} - P_i). \]

Which if we take \( Q_{0,i} = P_i \), can be written as:

\[ Q_{1,i} = n(Q_{0,i+1} - Q_{0,i}). \]
Solution

This suggests the following candidate:

\[ \gamma^{(j)}(t) = \sum_{i=0}^{n-j} Q_{j,i} \theta_{i,n-j}(t), \]

\[ Q_{j,i} = (n - (j - 1))(Q_{j-1,i+1} - Q_{j-1,i}), \]

With:

\[ j = 0 \Rightarrow Q_{0,i} = P_i. \]
Solution

We now prove by induction. For $j = 1$ we must check if:

$$\gamma^{(1)}(t) = \sum_{i=0}^{n-1} Q_{1,i} \theta_{i,n-1}(t).$$
Solution

Which indeed holds, since:

\[ Q_{1,i} = (n - (1 - 1))(Q_{1-1,i+1} - Q_{1-1,i}) = n(Q_{0,i+1} - Q_{0,i}) = n(P_{i+1} - P_i). \]

And this is exactly the formula that was shown in lectures (with *actual differentiation*).
Solution

The inductive step – assume correctness for $j$:

\[
\gamma^{(j)}(t) = \sum_{i=0}^{n-j} Q_{j,i} \theta_{i,n-j}(t),
\]

\[
Q_{j,i} = (n - (j - 1))(Q_{j-1,i+1} - Q_{j-1,i}),
\]
Solution

Then we must show that:

\[ \gamma^{(j+1)}(t) = \sum_{i=0}^{n-(j+1)} Q_{j+1,i} \theta_{i,n-(j+1)}(t), \]

\[ Q_{j+1,i} = (n - j)(Q_{j,i+1} - Q_{j,i}), \]
Solution

For this we use the \textit{correctness for first order} with \( n - j \) instead of \( n \), and differentiate the expression for \( \gamma^{(j)} \), which is correct by the inductive assumption.
Solution

\[ \gamma^{(j+1)}(t) = \frac{d}{dt} \left[ \sum_{i=0}^{n-j} Q_{j,i} \theta_{i,n-j}(t) \right] = \]

\[ = (n - j) \left( \sum_{i=0}^{n-j-1} (Q_{j,i+1} - Q_{j,i}) \theta_{i,n-j-1}(t) \right) \]

\[ = \sum_{i=0}^{n-(j+1)} Q_{j+1,i} \theta_{i,n-(j+1)}(t), \]

as required by the recursive formula.
Solution

Remark: recall the Casteljau algorithm:
Solution

Now, how does the differentiation look??
Solution

For $j = 1$:

Back to Casteljau from here

$\times n$
Solution

Question:
How will higher order differentiation look?
Q. 4

Find a *Bezier curve* passing through the origin at \( t = 0 \), with a *hodograph* given by the control points:

\[
Q_0 = (1,0), \quad Q_1 = (1,1), \quad Q_2 = (2,1)
\]
Solution

Recall that the *hodograph* of a Bezier curve \( \gamma \) of degree \( n \) is the curve:

\[
\frac{1}{n} \gamma'(t),
\]

thus we have:

\[
\frac{1}{n} \gamma'(t) = \sum_{i=0}^{2} \theta_{i,2}(t) Q_i.
\]
Solution

On the other hand — we know for the derivative:

\[ \gamma'(t) = n \sum_{i=0}^{n-1} [P_{i+1} - P_i] \theta_{i,n-1}(t), \]

or, for the hodograph:

\[ \frac{1}{n} \gamma'(t) = \sum_{i=0}^{n-1} [P_{i+1} - P_i] \theta_{i,n-1}(t). \]
Solution

And we know the control points of the hodograph – the $Q_i$’s. Therefore:

\[
P_1 - P_0 = Q_0 \\
P_2 - P_1 = Q_1 \\
P_3 - P_2 = Q_2
\]

(Question: Why can we say that?)
Solution

We now have three linear constraints and four unknowns (in each dimension!):

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{pmatrix}
=
\begin{pmatrix}
Q_0 \\
Q_1 \\
Q_2
\end{pmatrix}
\]
Solution

From the equations:

\[ P_1 - P_0 = Q_0 \]
\[ P_2 - P_1 = Q_1 \]
\[ P_3 - P_2 = Q_2 \]

we now can write:

\[ P_1 = Q_0 + P_0 \]
\[ P_2 = Q_1 + P_1 = Q_1 + Q_0 + P_0 \]
\[ P_3 = Q_2 + P_2 = Q_2 + Q_1 + Q_0 + P_0 \]
Solution

We still have one degree of freedom.

(Question: Why does this make sense?)

We still have to determine $P_0$. 
Solution

Since we are required that $\gamma(0) = (0,0)$:

$$P_0 = \gamma(0) = (0,0).$$

$$P_1 = Q_0 + P_0$$
$$P_2 = Q_1 + P_1 = Q_1 + Q_0 + P_0$$
$$P_3 = Q_2 + P_2 = Q_2 + Q_1 + Q_0 + P_0$$
Solution

Finally:

\[ P_1 = Q_0 + P_0 = (0,1) \]
\[ P_2 = Q_1 + Q_0 + P_0 = (1,1) + (0,1) = (1,2) \]
\[ P_3 = Q_2 + Q_1 + Q_0 + P_0 = (2,1) + (1,1) + (0,1) = (3,3). \]
Solution – Another Look

What did we do here?

In fact, this was the process of integrating a Bezier curve.
We were given:

\[ \gamma'(t) = 3 \sum_{i=0}^{2} \theta_{i,2}(t)Q_i. \]

which means:

\[ \gamma(t) = \int_{0}^{t} 3 \sum_{i=0}^{2} \theta_{i,2}(\tau)Q_i d\tau. \]
Remarks:

- By the vectors under the integral sign – we simply mean a shorthand for the same integral in each component.

- Note that the integration boundaries were chosen such that the initial conditions are met (definite integral!).
Solution – Another Look

Recall from lectures the integration formula for the Bernstein polynomials:

\[
\int_0^t \theta_{i-1,n-1}(\tau) d\tau = \frac{1}{n} \sum_{j=i}^{n} \theta_{j,n}(t)
\]
Solution – Another Look

Then for our Bezier curve, we had to find:

\[
\int_0^t \gamma'(\tau)d\tau = \int_0^t \sum_{i=0}^{2} 3\theta_{i,2}(\tau)Q_i d\tau
\]

\[
= \sum_{i=0}^{2} 3Q_i \int_0^t \theta_{i,2}(\tau)d\tau
\]

\[
= \sum_{i=0}^{2} 3Q_i \left[ \frac{1}{3} \sum_{j=i+1}^{3} \theta_{j,3}(t) \right] = \ldots
\]
Solution – Another Look

\[ Q_0 [\theta_{1,3}(t) + \cdots + \theta_{3,3}(t)] + Q_1 [\theta_{2,3}(t) + \theta_{3,3}(t)] + Q_2 [\theta_{3,3}(t)]. \]

Rearranging gives:

\[ \theta_{0,3}(t)(0,0) + \theta_{1,3}(t)Q_0 + \theta_{2,3}[Q_0 + Q_1] + \theta_{3,3}(t)[Q_0 + Q_1 + Q_2], \]
and indeed, in this form:
\[ \theta_{0,3}(t)(0,0) + \theta_{1,3}(t)Q_0 + \theta_{2,3}[Q_0 + Q_1] \\
+ \theta_{3,3}(t)[Q_0 + Q_1 + Q_2], \]

we see that the \( P_i \)'s we found are the same as here.
Points to Think About

- How does our solution change if the initial conditions were not at $t = 0$? Or not through the origin?

- We found a curve satisfying the demands. Are there any other ones?
Q.5

Find $\beta_1(t)$ and $\beta_2(t)$ obtained by subdivision of:

$$\gamma(t) = (1 - t)^2 (0,0) + 2t(1 - t)(1,1) + t^2 (3, -1)$$

at $t = 0.5$. 
Solution

We seek $\beta_1: [0,1] \rightarrow \mathbb{R}^2$ such that:

$$\beta_1(2t) = \gamma(t), \forall t \in [0,0.5],$$

and similarly, $\beta_2: [0,1] \rightarrow \mathbb{R}^2$ such that:

$$\beta_2(2t - 1) = \gamma(t), \forall t \in [0.5,1].$$
Solution

The general method for $\beta_1$ – derivatives of all orders must be equal for all $t$ values. Requiring this at $t = 0$ gives $n$ linear equations in $n$ unknowns (for each dimension).

A similar method for $\beta_2$ is used with $t = 1$. 
Solution

For $n = 2$ we choose a simpler method:

$$\beta_1(2t) = \gamma(t),$$

suggests the change of parameter:

$$r = 2t \Rightarrow t = \frac{r}{2}.$$
Solution

Now, we attempt to represent $\gamma$ in the parameter $r$:

$$\beta_1(r) = \gamma(t) = \gamma\left(\frac{r}{2}\right) = (1 - \frac{r}{2})^2 P_0 + 2\frac{r}{2}(1 - r) P_1 + \left(\frac{r}{2}\right)^2 P_2$$

$$= \left(1 - r + \frac{r}{2}\right)^2 P_0 + r(1 - r) + \frac{r}{2} P_1 + \frac{r^2}{4} P_2$$
Solution

\[ = [(1 - r)^2 + 2(1 - r)\frac{r}{2} + \frac{r^2}{4}]P_0 + (r(1 - r) + \frac{r^2}{2})P_1 + \frac{r^2}{4}P_2 = \ldots \]

Now we rearrange according to \( \theta_{i,2}(r) \):

\[ \ldots = (1 - r)^2P_0 + 2r(1 - r)\frac{P_0 + P_1}{2} + r^2 \frac{P_0 + 2P_1 + P_2}{4}. \]
Solution

Therefore, control points of $\beta_1$ are:

\[ Q_0 = P_0 = (0,0), \]
\[ Q_1 = \frac{P_0 + P_1}{2} = (0.5,0.5), \]
\[ Q_2 = \frac{P_0 + 2P_1 + P_2}{4} = (1.25,0.25). \]
Solution
Solution

Similarly, for $\beta_2$:

$$\beta_2(2t - 1) = \gamma(t),$$

suggests the change of parameter:

$$r = 2t - 1 \Rightarrow t = \frac{r + 1}{2}.$$
Solution

As before we rearrange $\gamma$ with the new parameter. This gives a symmetric result:

$$\beta_2(r) = \gamma(t) = \gamma\left(\frac{r+1}{2}\right)$$

$$= \left(1 - \frac{r+1}{2}\right)^2 P_0 + 2\frac{r+1}{2}(1 - \frac{r+1}{2})P_1$$

$$+ \left(\frac{r+1}{2}\right)^2 P_2 = \cdots$$
Solution

\[
(1 - r)^2 + 2(1 - r)\frac{r}{2} + \frac{r^2}{4} \]

\(P_0 + (r(1 - r) + \frac{r^2}{2})P_1 + \frac{r^2}{4}P_2 = \cdots\)

Again, rearrange according to \(\theta_{i,2}(r)\):

\[
\cdots = (1 - r)^2 \frac{P_0 + 2P_1 + P_2}{4} + 2r(1 - r) \frac{P_1 + P_2}{2} + r^2P_2.
\]
Solution

Therefore, control points of \( \beta_2 \) are:

\[
A_0 = \frac{P_0 + 2P_1 + P_2}{4} = (1.25,0.25),
\]

\[
A_1 = \frac{P_1 + P_2}{2} = (2,0),
\]

\[
A_2 = P_2 = (3, -1).
\]
Solution
Solution
Solution
Solution
Remarks:

- Subdivision can be performed at any parameter value $t$.
- The general solution: Compare derivatives of all orders at end points of new curves.
- Result: *Control points of the new curves are the left and right sides* of the recursive evaluation tree!
Solution

Control Points of $\beta_1$

$Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow Q_3$

$Q_1 \rightarrow Q_2 \rightarrow Q_3$

$Q_2 \rightarrow Q_3$

Control Points of $\beta_2$

$Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow Q_3$

$Q_1 \rightarrow Q_2 \rightarrow Q_3$

$Q_2 \rightarrow Q_3$

$Q_3$

$t$ and $1-t$ relationships between points.
Solution

Control Points of $\beta_1$

Example:

$\gamma'(0) = n(Q_1 - Q_0)$

$\beta_1'(0) = n(Q_1^1 - Q_0) = n(tQ_1 + (1 - t)Q_0 - Q_0) = n(tQ_1 - tQ_0) = tn(Q_1 - Q_0)$