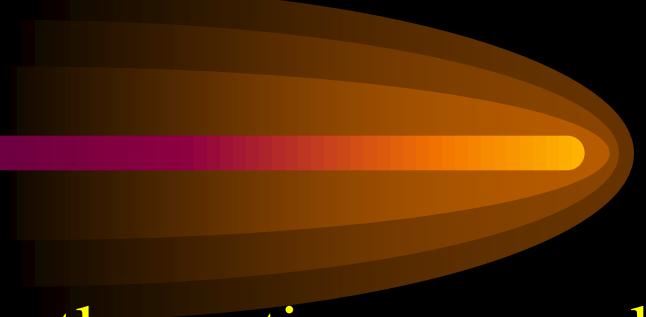


# Osculating Sphere to a Parametric Curve

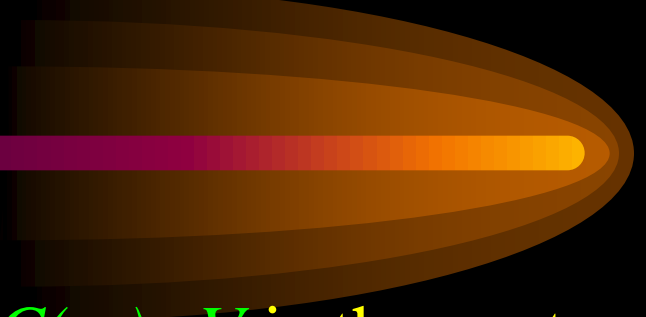


- Let  $C(s)=(x(s),y(s),z(s))$  be a sufficiently continuous regular parametric curve parameterized by an arc length parameter  $s$
- Consider  $C(s_0)$  and  $C(s_0 + \delta)$
- Taylor expansion:

$$C(s_0 + \delta) = C(s_0) + \delta T(s_0) + \frac{\delta^2}{2} C''(s_0) + \frac{\delta^3}{6} C'''(s_0) + \dots$$

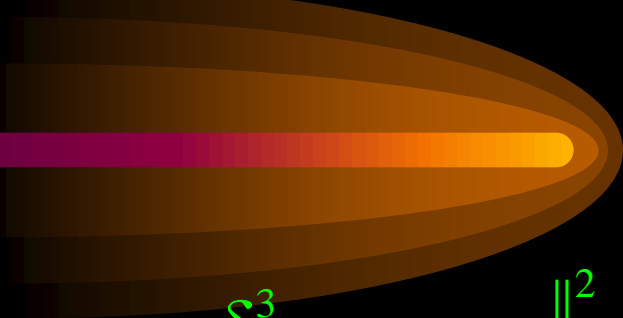
$$= C(s_0) + \delta T(s_0) + \frac{\delta^2}{2} k(s_0)N(s_0) + \frac{\delta^3}{6} C'''(s_0) + \dots$$

# Osculating Sphere to a Parametric Curve



- Let  $V$  be a vector of length  $r$  so that  $C(s_0)+V$  is the center of a sphere tangent to  $C(s_0)$ .
- A point  $P$  is on the sphere iff  $\|C(s_0)+V - P\|=r$ .
- Define  $\Delta = \|C(s_0)+V - P\|^2 - r^2$
- Assume  $P = C(s_0 + \delta)$ . We want to find the conditions under which  $C(s_0 + \delta)$  is closest to the sphere.

# Osculating Sphere to a Parametric Curve



$$\begin{aligned}
 \Delta &= \|C(s_0) + V - C(s_0 + \delta)\|^2 - r^2 \\
 &= \left\| C(s_0) + V - C(s_0) - \delta T(s_0) - \frac{\delta^2}{2} k(s_0) N(s_0) - \frac{\delta^3}{6} C'''(s_0) \dots \right\|^2 - r^2 \\
 &= \left\| V - \delta T(s_0) - \frac{\delta^2}{2} k(s_0) N(s_0) - \frac{\delta^3}{6} C'''(s_0) \dots \right\|^2 - r^2 \\
 &= -2\delta T(s_0)V + \delta^2 (T^2(s_0) - V k(s_0) N(s_0)) \\
 &\quad - \delta^3 \left( \frac{1}{3} V C'''(s_0) - T(s_0) k(s_0) N(s_0) \right) - \dots
 \end{aligned}$$

# Osculating Sphere to a Parametric Curve

$$\begin{aligned}\Delta &= -2\delta T(s_0)V + \delta^2(1 - Vk(s_0)N(s_0)) - \delta^3\left(\frac{1}{3}VC'''(s_0)\right) - \dots \\ &= -2\delta T(s_0)V + \delta^2(1 - Vk(s_0)N(s_0)) \\ &\quad - \delta^3\frac{1}{3}V \\ &\quad \left(-k^2(s_0)T(s_0) + k'(s_0)N(s_0) + k(s_0)\tau(s_0)B(s_0)\right) - \dots\end{aligned}$$

# Osculating Sphere to a Parametric Curve

$$\begin{aligned}C'''(s) &= (C''(s))' \\ &= (k(s)N(s))' \\ &= k'(s)N(s) + k(s)(-k(s)T(s) + \tau(s)B(s)) \\ &= -k^2(s)T(s) + k'(s)N(s) + k(s)\tau(s)B(s)\end{aligned}$$

- If  $Vk(s_0)N(s_0)=1$  (or in other words  $V = N(s_0)/k(s_0)$ ) then

$$\Delta = -\delta^3 \frac{1}{3} V \left( -k^2(s_0)T(s_0) + k'(s_0)N(s_0) + k(s_0)\tau(s_0)B(s_0) \right) + \dots$$

# Osculating Sphere to a Parametric Curve

- Employing  $V N(s_0) = 1/k(s_0)$  and  $VT(s_0) = 0$  we have

$$\Delta = \delta^3 \frac{1}{3} \left( \frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0) \right) + \dots$$

- An additional requirement

$$\frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0) = 0$$

provides contact degree four

# Osculating Sphere to a Parametric Curve

$$\frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0) = 0 \text{ implies}$$

$$V = \frac{1}{k(s_0)}N(s_0) - \frac{k'(s_0)}{k^2(s_0)\tau(s_0)}B(s_0) = 0$$

- Question: What is the intersection of the osculating sphere and the osculating plane ?
- Hint: Let  $W = \frac{1}{k(s_0)}N(s_0)$ . What is  $(VN(s_0))N(s_0)$  in terms of  $W$ ?