Osculating Sphere to a Parametric Curve

- Let $C(s)=(x(s),y(s),z(s))$ be a sufficiently continuous regular parametric curve parameterized by an arc length parameter s
- Consider $C(s_0)$ and $C(s_0+\delta)$
- Taylor expansion:

$$C(s_0+\delta)=C(s_0)+\delta T(s_0)+\frac{\delta^2}{2}C''(s_0)+\frac{\delta^3}{6}C'''(s_0)+\dots$$

$$=C(s_0)+\delta T(s_0)+\frac{\delta^2}{2}k(s_0)N(s_0)+\frac{\delta^3}{6}C'''(s_0)+\dots$$

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- Let V be a vector of length r so that $C(s_0)+V$ is the center of a sphere tangent to $C(s_0)$.
- A point P is on the sphere iff $||C(s_0)+V-P||=r$.
- Define $\Delta = \left\| C(s_0) + V - P \right\|^2 - r^2$
- Assume $P = C(s_0+\delta)$. We want to find the conditions under which $C(s_0+\delta)$ is closest to the sphere.

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$$\Delta = \left\| C(s_0) + V - C(s_0 + \delta) \right\|^2 - r^2$$

$$= \left\| C(s_0) + V - C(s_0) - \delta T(s_0) - \frac{\delta^2}{2} k(s_0) N(s_0) - \frac{\delta^3}{6} C'''(s_0) \ldots \right\|^2 - r^2$$

$$= \left\| V - \delta T(s_0) - \frac{\delta^2}{2} k(s_0) N(s_0) - \frac{\delta^3}{6} C'''(s_0) \ldots \right\|^2 - r^2$$

$$= -2\delta T(s_0) V + \delta^2 \left(T^2(s_0) - V k(s_0) N(s_0) \right)$$

$$- \delta^3 \left(\frac{1}{3} V C'''(s_0) - T(s_0) k(s_0) N(s_0) \right) - \ldots$$

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$$\Delta = -2\delta T(s_0)V + \delta^2\left(1 - Vk(s_0)N(s_0)\right) - \delta^3\left(\frac{1}{3}VC'''(s_0)\right) - \ldots$$

$$= -2\delta T(s_0)V + \delta^2\left(1 - Vk(s_0)N(s_0)\right)$$

$$-\delta^3\frac{1}{3}V$$

$$\left(-k^2(s_0)T(s_0) + k'(s_0)N(s_0) + k(s_0)\tau(s_0)B(s_0)\right) - \ldots$$

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$$C'''(s) = \left(C''(s)\right)'$$

$$= \left(k(s)N(s)\right)'$$

$$= k'(s)N(s) + k(s)\left(-k(s)T(s) + \tau(s)B(s)\right)$$

$$= -k^2(s)T(s) + k'(s)N(s) + k(s)\tau(s)B(s)$$

- If $Vk(s_0)N(s_0)=1$ (or in other words $V= N(s_0)/k(s_0)$) then

$$\Delta = -\delta^3 \frac{1}{3} V\left(-k^2(s_0)T(s_0) + k'(s_0)N(s_0) + k(s_0)\tau(s_0)B(s_0)\right) + \ldots$$

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- Employing $V\,N(s_0)=1/k(s_0)$ and $VT(s_0)=0$ we have

$$\Delta = \delta^3 \frac{1}{3}\left(\frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0)\right) + \ldots$$

- An additional requirement

$$\frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0) = 0$$

provides contact degree four

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$$\frac{k'(s_0)}{k(s_0)} + k(s_0)\tau(s_0)V \cdot B(s_0) = 0 \text{ implies}$$

$$V = \frac{1}{k(s_0)} N(s_0) - \frac{k'(s_0)}{k^2(s_0)\tau(s_0)} B(s_0) = 0$$

- Question: What is the intersection of the osculating sphere and the osculating plane ?
- Hint: Let $W = \frac{1}{k(s_0)} N(s_0)$. What is $(VN(s_0))N(s_0)$ in terms of W?