Computer Aided Geometric Design

The Bézier curve-curve intersection problem

- How the geometric properties of Bézier curves are used in CCI solving
The convex hull property

Recall: a Bézier curve is contained inside the convex hull of its control mesh.
The convex hull property

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The convex hull property

Useful for non-intersection testing:
For curves $c_1(t), c_2(r)$, if the convex hulls of control meshes of $c_1$ and $c_2$ are disjoint, then $c_1$ and $c_2$ do not intersect.
The convex hull property

If the convex hulls of the control meshes do intersect then $c_1$ and $c_2$ may intersect. How many times?
Testing for multiple intersections

The hodograph of \( c(t) \) is the curve of tangent directions of \( c \). Equals to \( c'(t) \).

Bound the set of tangent directions in a pair of cones.
Testing for multiple intersections

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Bound the set of tangent directions in a **pair of cones**.
Testing for multiple intersections

If we position the pair of cones on any point on \( c(t) \), the entire curve will be contained in the cones.

Proof: Position cones at \( c(t_0) \) and assume (falsely) \( c(t_1) \) is outside the cones.

- There is \( t_2 \) such that \( \pm c'(t_2) = c(t_1) - c(t_0) \).

But the cones where constructed such that they contain all of \( c'(t) \) (including \( c'(t_2) \)). This is a contradiction. ■
Testing for multiple intersections

According to this theorem, if the cones of tangent directions of \( c_0 \) and \( c_1 \) do not overlap (except for the apex), then \( c_0 \) and \( c_1 \) intersect at most once.
Intermediate summary

- Convex hull test for discriminating between cases where the curves have no intersections vs. may have intersections.
- Cones of tangent directions test for discriminating between cases where the curves have at most one intersection vs. may any number of intersections.
CCI solution complete algorithm

- Convex hull test
  - No intersections
    - Stop (no intersections)
  - May intersect
    - At most one intersection
      - Try solving numerically
        - Fail
          - Subdivide
          - Any number of intersections
            - Stop and return result
        - Succeed
          - Stop and return result
  - Tangent cones test
    - May intersect
      - At most one intersection
        - Try solving numerically
          - Fail
            - Subdivide
            - Any number of intersections
              - Stop and return result
          - Succeed
            - Stop and return result