The Interpolation Problem

- The general interpolation problem
  - We have a set of abscissas $\{u_i\}_{i=0}^{s}$ and a function $f(x)$
  - Target: to find a polynomial $P(x)$ such that $P(u_i) = f(u_i),\ i = 1, \ldots, s$
  - Observation: the function $f(x)$ is not “critical”, one can substitute it by the set of values $\{p_i\}_{i=1}^{s},\ f(u_i) = p_i$.

Specific Interpolation Problems

- We have a set of abscissas $\{u_i\}_{i=0}^{s}$ and a function $f(x)$
- Target: to find a polynomial $P(x)$ such that $P(u_i) = f(u_i)$,
- $P'(u_i) = f'(u_i), \quad i = 1, \ldots, s$. This is the classical Hermite interpolation.
- Target: to find $P(x)$ such that in each abscissa there is a certain degree of contact between $f(x)$ and $P(x)$. This is the general Hermite interpolation.
- Observation: once again the function $f(x)$ is not "critical".

B-spline Functions Interpolation Problems

- We have a set of knots $\{u_i\}_{i=0}^{s}$
- and a function $c(t)=\sum_{i=0}^{s+2} P_i B_{i,3}(t)$
- Requirements:
  - $c(u_i)=\sum_{i=0}^{s+2} P_i B_{i,3}(u_i)=p_i$, $s+1$ constraints
  - $c'(u_i-\varepsilon)=c'(u_i+\varepsilon)$, s-1 constraints
  - $c''(u_i-\varepsilon)=c''(u_i+\varepsilon)$, s-1 constraints
  - Two additional requirements: $c'(u_0)=p'_0$ and $c'(u_s)=p'_s$

Complete Cubic Interpolation with B-splines

- B-spline interpolants are polynomials by parts
- There are discontinuities at knots
- Requirements:
  - Degree 3
  - Continuity $C^{(2)}$
- The abscissas $\{u_i\}_{i=0}^{s}$ are knots
- First and immediate diagnosis: the internal knots $u_{i=1}^{s-1}$ have multiplicity 1

Complete Cubic Interpolation with B-splines

- The B-spline curve: $c(t) = \sum_{i=0}^{s+2} P_i B_{i,3}(t)$
- We have to interpolate s+1 values to interpolate
- We have $s+3$ degrees of freedom P_i

Complete Cubic Interpolation with B-splines

- In order to interpolate (u_0, p_0) and (u_s, p_s) we choose open end conditions

$$t_j = u_0,\ j = 0,1,2,3, \quad t_j = u_{j-3},\ j = 4,\ldots,s+2, \quad t_j = u_s,\ j = s+3,\ldots,s+6$$

- The end points are: $p_0 = P_0$ and $p_s = P_{s+2}$
- In general: $c(u_j) = \sum_{i=0}^{s+2} P_i B_{i,3}(u_j) = p_j$
- Two additional constraints: $c'(u_j) = \sum_{i=0}^{s+2} P_i B'_{i,3}(u_j) = p'_j$

Complete Cubic Interpolation with B-splines

- At $u_0 = t_0 = t_1 = t_2 = t_3$ only $B'_{0,3} \neq 0$ and $B'_{1,3} \neq 0$
- Moreover, $\sum_{i=0}^{3} B'_{i,3} = 0$
- Therefore, $B'_{0,3} + B'_{1,3} = 0$
- Similarly, $B'_{s+1,3} + B'_{s+2,3} = 0$

Complete Cubic Interpolation with B-splines

- Matriceal system of equations

$$B\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_s \\ P_{s+1} \\ P_{s+2} \end{pmatrix} = \begin{pmatrix} p_0 \\ p_0' \\ p_1 \\ \vdots \\ p_{s-1} \\ p_s' \\ P_s \end{pmatrix}$$

Complete Cubic Interpolation with B-splines

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ B'_{0,3} & B'_{1,3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & B_{1,3} & B_{2,3} & B_{3,3} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & B_{s,3} & B_{s+1,3} & B_{s+2,3} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & B'_{s+1,3} & B'_{s+2,3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$