Affine Knots Changing

- Consider the knot sequence $t = [t_0, t_1, \ldots, t_n]$ and the order k
- Define $\tau = \tau_0, \tau_1, \ldots, \tau_n$ by setting $\tau_i = at_i + b$, $a>0$.
- We will prove that $B_{i,k,t}(x) = B_{i,k,\tau}(ax + b)$
- This property is called the affine change of knots

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- We will prove the affine change of knots property by induction on the order k
- For $k = 1$

$$B_{i,1,\tau}(ax+b)=1=B_{i,1,t}(x)$$

- Assume the affine change of knots property is true for Bsplines of order k-1

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$$
\begin{aligned}
&B_{i,k,\tau}(ax+b) \\
&= \frac{(ax+b-\tau_i)}{\tau_{i+k-1}-\tau_i} B_{i,k-1,\tau}(ax+b) + \frac{(\tau_{i+k}-ax-b)}{\tau_{i+k}-\tau_{i+1}} B_{i+1,k-1,\tau}(ax+b) \\
&= \frac{(ax+b-\tau_i)}{\tau_{i+k-1}-\tau_i} B_{i,k-1,\tau}(ax+b) + \frac{(\tau_{i+k}-ax-b)}{\tau_{i+k}-\tau_{i+1}} B_{i+1,k-1,\tau}(ax+b) \\
&= \frac{(ax+b-at_i-b)}{at_{i+k-1}+b-at_i-b} B_{i,k-1,\tau}(ax+b) + \frac{(at_{i+k}+b-ax-b)}{at_{i+k}+b-at_{i+1}-b} B_{i+1,k-1,\tau}(ax+b) \\
&= \frac{a(x-t_i)}{a(t_{i+k-1}-t_i)} B_{i,k-1,\tau}(ax+b) + \frac{a(t_{i+k}-x)}{a(t_{i+k}-t_{i+1})} B_{i+1,k-1,\tau}(ax+b) \\
&= \frac{(x-t_i)}{(t_{i+k-1}-t_i)} B_{i,k-1,t}(x) + \frac{(t_{i+k}-x)}{(t_{i+k}-t_{i+1})} B_{i+1,k-1,t}(x) \; = B_{i,k,t}(x)
\end{aligned}
$$

Affine Knots Changing

- We have $B_{i,k,\tau}(ax+b) = B_{i,k,t}(x)$, and therefore, the affine change of knots property holds for any k.