Computer Aided Geometric Design

Class Exercise #10

- Differential Geometry of Surfaces – Part I

Q1.

Calculate the length of the parametric curve:

$$c(t) = \left(0, t, \sqrt{1-t^2}\right), t \in [0,1]$$

in two ways:

1. As a curve in $\mathbb{R}^3$.
2. As a curve on a sphere, using the First Fundamental Form.

Solution

For part 1 we ignore the fact that the curve is spherical and treat it as a curve in $\mathbb{R}^3$:

$$L = \int_0^1 \|c'(t)\| dt$$

Solution

Now:

$$c'(t) = \left(0, 1, \frac{-t}{\sqrt{1 - t^2}}\right)$$

$$\|c'(t)\| = \sqrt{0 + 1 + \frac{t^2}{1 - t^2}} = \sqrt{\frac{1}{1 - t^2}}$$

Solution

Therefore:

$$L = \int_0^1 \|c'(t)\| dt = \int_0^1 \sqrt{\frac{1}{1-t^2}}\, dt$$

$$= arcsin(t)\Big|_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Solution

For part 2 we ignore the fact that the curve is in $\mathbb{R}^3$, and use the fact that it is in on the (parameterized) half-sphere:

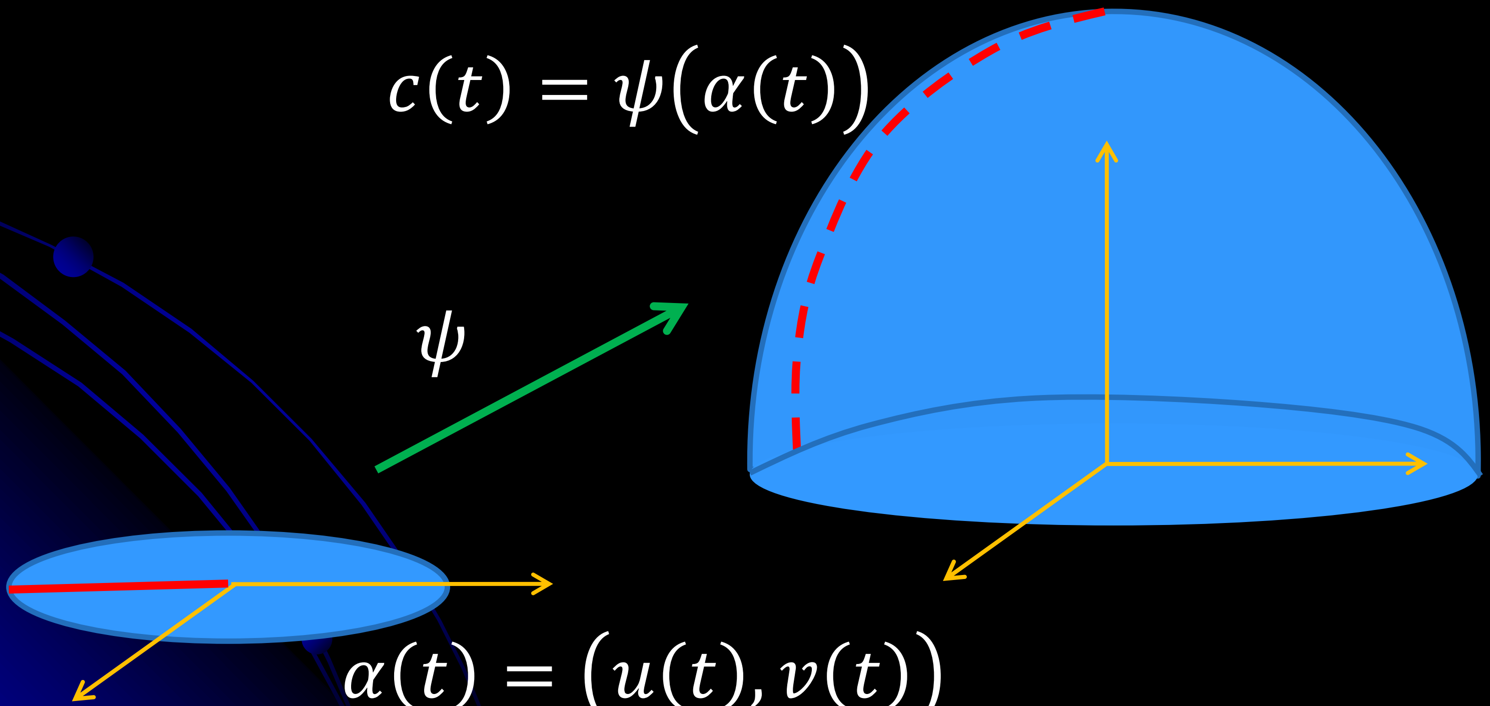
$$\psi(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right),$$

$$(u,v) \in \textit{unit disc in the plane}.$$

Solution

A curve on a parameterized surface is the restriction of the parameterization ψ to a curve in the parameter space:

$$c(t) = \psi\big(\alpha(t)\big)$$



$$\alpha(t) = \big(u(t), v(t)\big)$$

Solution

Recall the First Fundamental Form I_p defined on tangent vectors w to the surface S at a point $p \in S$:

$$w \mapsto ||w||^2 = \langle w, w \rangle$$

Where $\langle \cdot,\cdot \rangle$ is the inner product on the tangent space.

Solution

(Remark:

- In this case $\langle \cdot,\cdot \rangle$ is inherited from $\mathbb{R}^3$, but this need not be the case in abstract manifolds)

Solution

In lectures you saw that using the chain rule, the following coefficients are obtained:

$$c'(t) = \frac{d}{dt}\left[\psi\big(\alpha(t)\big)\right] \Rightarrow$$

$$\langle c'(t), c'\,(t)\rangle = Eu'^2 + 2Fu'v' + Gv'^2$$

When:

$$E = E(u,v) = \langle \psi_u, \psi_u\rangle$$

$$F = F(u,v) = \langle \psi_u, \psi_v\rangle$$

$$G = G(u,v) = \langle \psi_v, \psi_v\rangle$$

Solution

Therefore, the length of c as a curve in the surface parameterized by ψ is:

$$\int_0^1 \|c'(t)\| dt = \int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt$$

When all functions involved are evaluated at t, namely: $E\big(u(t), v(t)\big)$, etc.

Solution

The coefficients in our parameterization are:

$$E = \langle \psi_u, \psi_u \rangle$$

$$= \left\langle \left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right), \left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right) \right\rangle$$

$$= 1 + \frac{u^2}{1-u^2-v^2} = \frac{1-v^2}{1-u^2-v^2}$$

Solution

The coefficients in our parameterization are:

$$F = \langle \psi_u, \psi_v \rangle$$
$$= \left\langle \left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right), \left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right) \right\rangle$$
$$= \frac{u-v}{1-u^2-v^2}$$

Solution

The coefficients in our parameterization are:

$$G = \langle \psi_v, \psi_v \rangle$$

$$= \left\langle \left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right), \left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right) \right\rangle$$

$$= 1 + \frac{v^2}{1-u^2-v^2} = \frac{1-u^2}{1-u^2-v^2}$$

Solution

and:

$$\int_0^1 \|c'(t)\| dt = \int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\, dt$$

$$= \int_0^1 \sqrt{E \cdot 0 + 2F \cdot 0 \cdot 1 + G \cdot 1}\, dt = \int_0^1 \sqrt{G}\, dt$$

$$= \int_0^1 \sqrt{\frac{1-0}{1-t^2}}\, dt = \frac{\pi}{2}$$

Final Remarks

- In local coordinates, the First Fundamental Form $w \mapsto \|w\|^2$ is the quadratic form:

$$w^t \boldsymbol{G} w$$

Where:

$$w = \begin{pmatrix} u' \\ v' \end{pmatrix},$$

$$\boldsymbol{G} = \boldsymbol{G}(u, v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Final Remarks

- Can you think of a point of view for the first solution that also uses:

$$L = \int_{t=a}^{b} \sqrt{w^t \boldsymbol{G} w}\, dt \qquad ?$$

Q2.

Consider a surface S parameterized by $\psi(u, v)$ and denote:

$$\boldsymbol{G} = \boldsymbol{G}(u, v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Prove that at a point $p \in S$, the length of the normal to S at p is given by $\sqrt{det(\boldsymbol{G})}$.

Solution

The normal is given by:

$$\psi_u \times \psi_v$$

and the length satisfies:

$$\|\psi_u \times \psi_v\|^2 = \|\psi_u\|^2 \|\psi_v\|^2 \sin^2 \theta$$

Solution

Where θ is the angle between ψ_u and ψ_v.

Now:

$$\|\psi_u \times \psi_v\|^2 = \|\psi_u\|^2 \|\psi_v\|^2 \sin^2\theta$$
$$= \|\psi_u\|^2 \|\psi_v\|^2 (1 - \cos^2\theta)$$
$$= \|\psi_u\|^2 \|\psi_v\|^2 - \|\psi_u\|^2 \|\psi_v\|^2 \cos^2\theta =$$

Solution

Writing this using inner products:

$$\|\psi_u\|^2\|\psi_v\|^2 - \|\psi_u\|^2\|\psi_v\|^2\cos^2\theta$$
$$= \langle\psi_u,\psi_u\rangle\langle\psi_v,\psi_v\rangle - \langle\psi_u,\psi_v\rangle^2 = EG - F^2$$
$$= det(\boldsymbol{G}).$$

Final Remarks

- This means that the first fundamental form enables us to compute the surface area, using only the parameterization via $\sqrt{det(\boldsymbol{G})}$.

- This fact remains for any abstract manifold in higher dimensions. $\boldsymbol{G}$ is then called a Riemannian metric, and $\sqrt{det(\boldsymbol{G})}$ is called the Riemannian Volume form.

Q3.

Use question 2 to compute the surface area of a torus with radius $a > b$.

Solution

First we need a parameterization:

$$\psi(u,v) = \begin{pmatrix} \big(a + b\cos(v)\big)\cos(u) \\ \big(a + b\cos(v)\big)\sin(u) \\ bsin(v) \end{pmatrix}$$

$$(u,v) \in [0,2\pi] \times [0,2\pi].$$

(Why?)

Solution

The First Fundamental Form is computed by differentiation:

$$\psi_v = \begin{pmatrix} -b\sin(v)\cos(u) \\ -b\sin(v)\sin(u) \\ b\cos(v) \end{pmatrix}$$

$$\psi_u = \begin{pmatrix} -\big(a + b\cos(v)\big)\sin(u) \\ \big(a + b\cos(v)\big)\cos(u) \\ 0 \end{pmatrix}$$

Solution

Which gives:

$$E = \langle \psi_u, \psi_u \rangle = \left(a + b\cos(v)\right)^2$$

$$F = \langle \psi_u, \psi_v \rangle = 0$$

$$G = \langle \psi_v, \psi_v \rangle$$
$$= b^2 \sin^2 v \cos^2 u + b^2 \sin^2 v \sin^2 u$$
$$+ b^2 \cos^2 u = b^2$$

Solution

Now:

$$S = \int_0^{2\pi}\int_0^{2\pi} \|N(u,v)\|\,dudv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\,dudv$$

$$= \int_0^{2\pi}\int_0^{2\pi} b\big(a + b\cos(u)\big)\,dudv =$$

Solution

$$= 2\pi \int_{0}^{2\pi} b\left(a + b\cos(u)\right) du$$

$$= 2\pi\left[b\left(au + bsin(u)\right)\right]_0^{2\pi} = 4\pi^2 ab$$

Final Remarks

- How is this related to the surface area of a cylinder?

(Hint: how can one get a torus from a cylinder?)

- What is the geometric interpretation of $F = 0$?