Computer Science Department
TECHNION Israel Institute of Technology

Kinematics of linkages

“Computer Aided Geometric Design”

Final Presentation

Computer Science Department TECHNION Israel Institute of Technology

Agenda

- Introduction to mechanisms
- Mapping of the motion into algebraic constraints.
- Examples of linkages
- Major research and development
  - “Geometric Constraint Solver using Multivariate Rational Spline Functions” Gershon Elber, Myung-Soo Kim.
  - “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.
  - “Precise contact motion planning for deformable planar curved shapes” Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

Introduction to mechanisms

Kinematic Mechanism

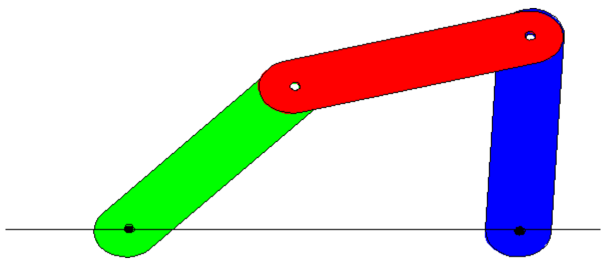
- A mechanism is a device to transform one motion into another.
- Comprised of rigid bodies connected such that each one moves with respect to another.
- The connection is a joint between two members permitting a particular kind of motion.
- The joints between links are modeled as providing ideal movement, pure rotation or sliding.

source:
https://www.google.com/url?sa=i&url=https%3A%2F%2Fen.wikipedia.org%2Fwiki%2FLinkage_(mechanical)&psig=AOvVaw1T2vVcsb82GNPemrHYo_w9&ust=1643014952376000&source=images&cd=vfe&ved=0CAsQjRxqFwoTCNib6OnBx_UCFQAAAAAdAAAAABAE

Mapping of the motion into algebraic constraints.

Representation of a Mechanism

- A kinematic mechanism M = { E, C }, contains:
  - E - a set of elements from which the mechanism is built.
  - C - a set of constraints among the element.
- Under motion of the mechanism, the constraints C should be preserved.



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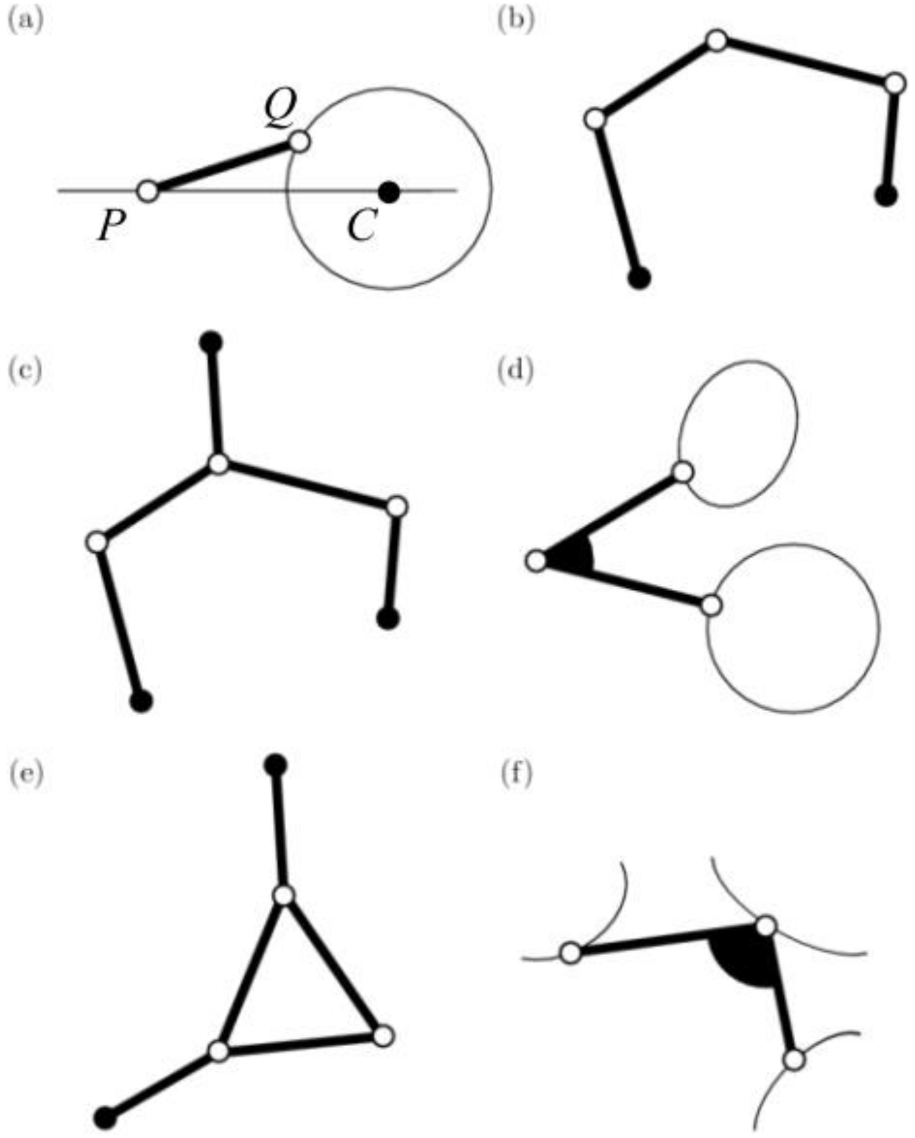
The elements of a mechanism

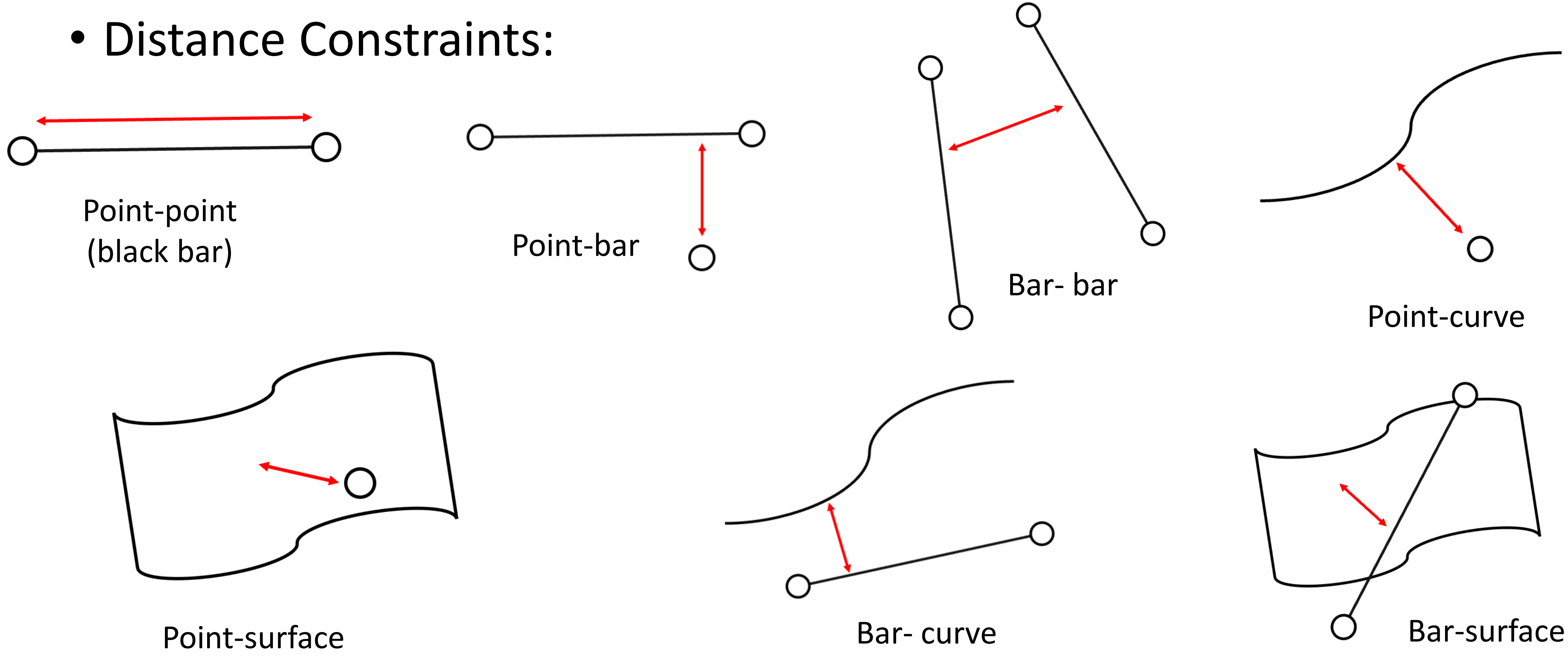


- The basic build block of the mechanism is a 2D or a 3D point.
- The point types:
  - **An anchored point** - point that lies fixed and does not change its position during the motion.
  - **Point on a Curve** - point that can move while its trajectory is constrained to a curve.
  - **Point on a Surface** - point that can move while its trajectory is constrained to be on a surface.
  - **A free point** - point that can move in any direction.
- Kinematic bars and kinematic faces are defined as pairs and triplets of kinematic points, respectively.

source: source:Kinematic Simulation of Planar and Spatial Mechanisms using a polynomial constraint solver" Michael Barton, Nadav Shragai, Gershon Elber.

The Constraints in a Mechanism

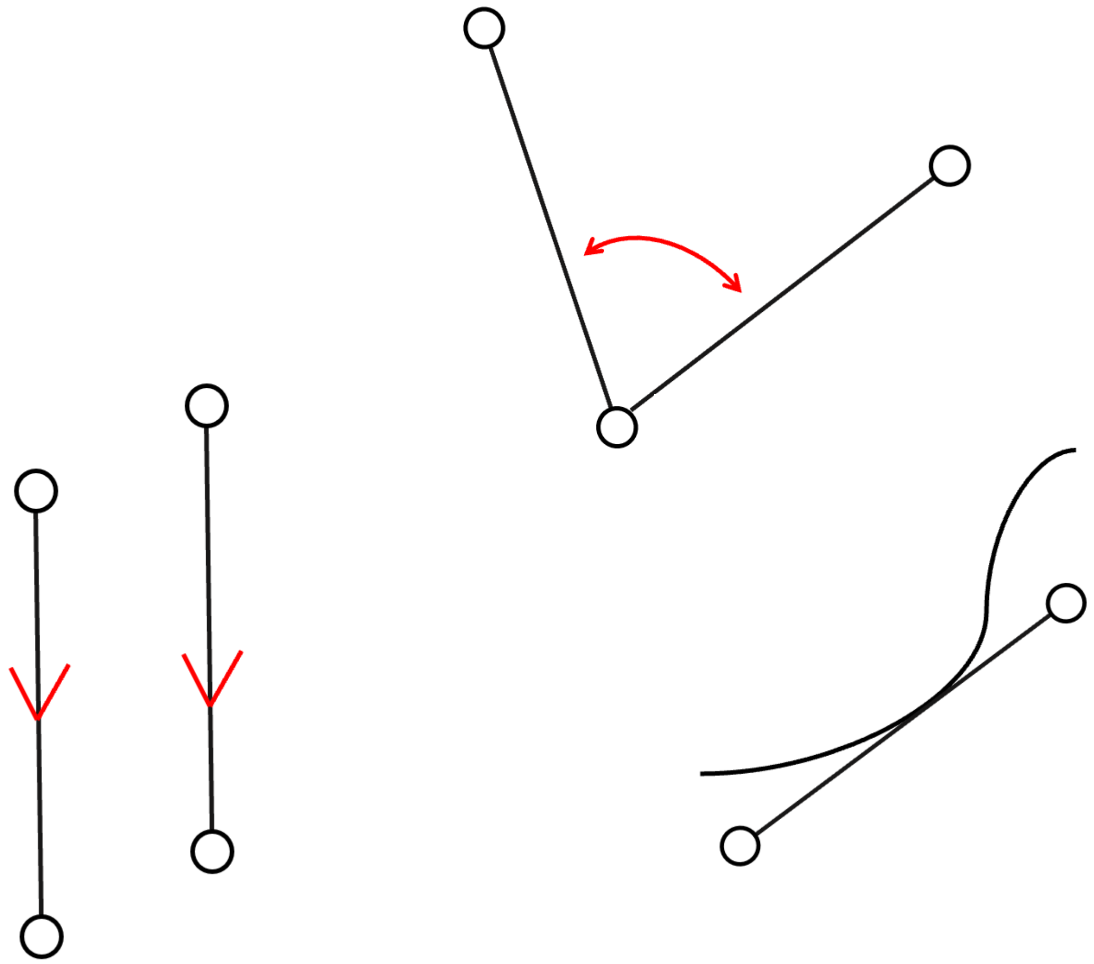
- Distance Constraints:



source: source:Kinematic Simulation of Planar and Spatial Mechanisms using a polynomial constraint solver" Michael Barton, Nadav Shragai, Gershon Elber.

The Constraints in a Mechanism

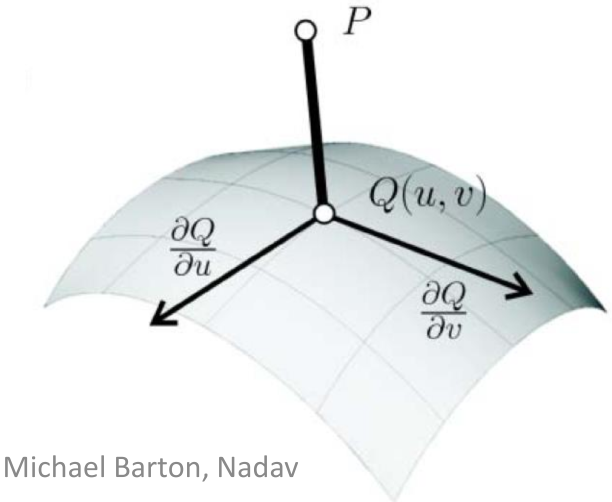
- Angular Constraints:
  - bar-bar
  - bar-plane,
- Tangency:
  - bar--curve
  - bar—surface
  - face-surface
- Parallelism
  - Bar-bar



source: source:Kinematic Simulation of Planar and Spatial Mechanisms using a polynomial constraint solver" Michael Barton, Nadav Shragai, Gershon Elber.

Examples

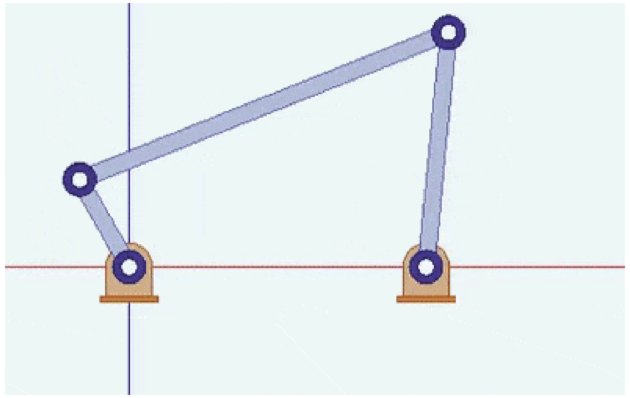
- Distance preserving constraints d between points P and Q
  - $||P - Q||^2 - d^2 = 0$
- Angle constraints between two bars PQ and RT
  - $\frac{\langle P-Q, R-T\rangle^2}{||P-Q||^2 ||R-T||^2} - \cos^2(\alpha) = 0$
- Point-surface distance between point P and surface $Q(u, v)$
  - $\begin{cases} ||P - Q|| - d^2 = 0 \\ \left\langle \frac{\partial Q}{\partial u}, P - Q \right\rangle = 0 \\ \left\langle \frac{\partial Q}{\partial v}, P - Q \right\rangle = 0 \end{cases}$



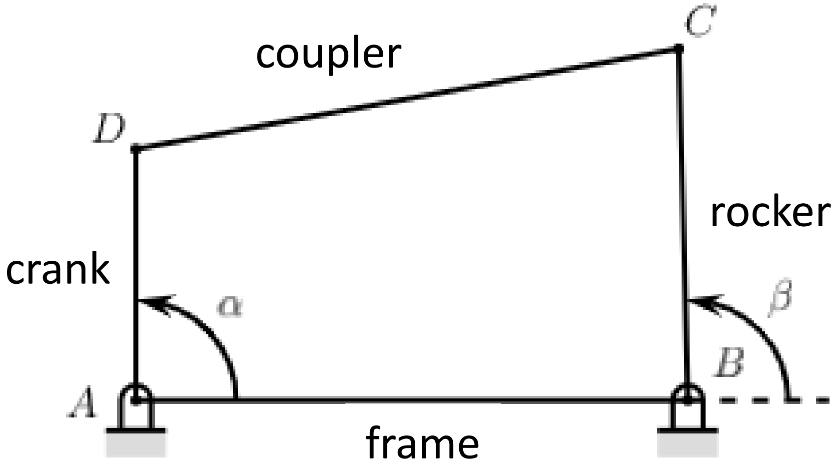
source: source:Kinematic Simulation of Planar and Spatial Mechanisms using a polynomial constraint solver" Michael Barton, Nadav Shragai, Gershon Elber.

Examples of linkages

Four-bar linkage



- A planar four-bar linkage consists of four rigid rods in the plane connected by pin joints. We call the rods:
  - Frame link: fixed to anchor pivots AA and BB.
  - crank link: driven by input angle α.
  - rocker link: gives output angle β.
  - Rocker link: connects the two moving pins C and D



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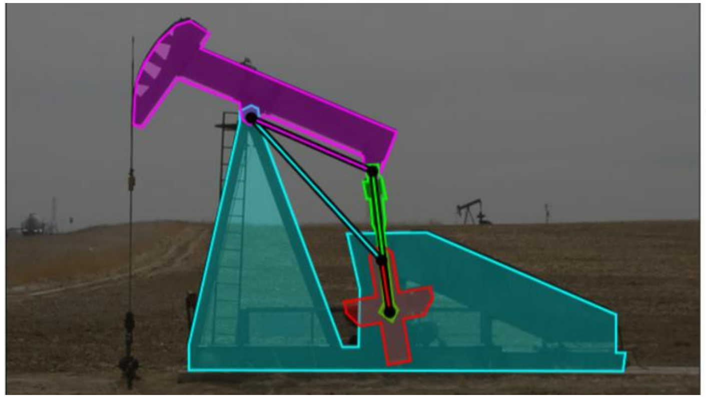
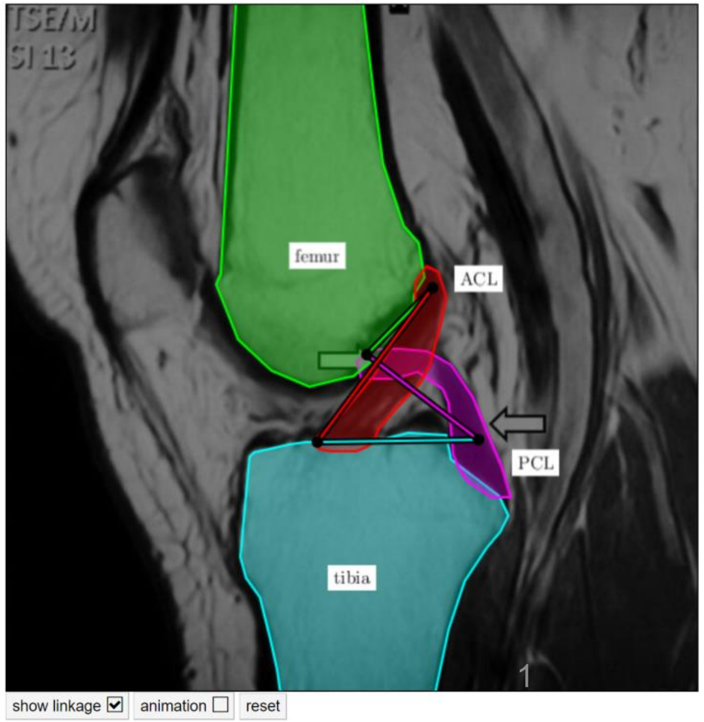
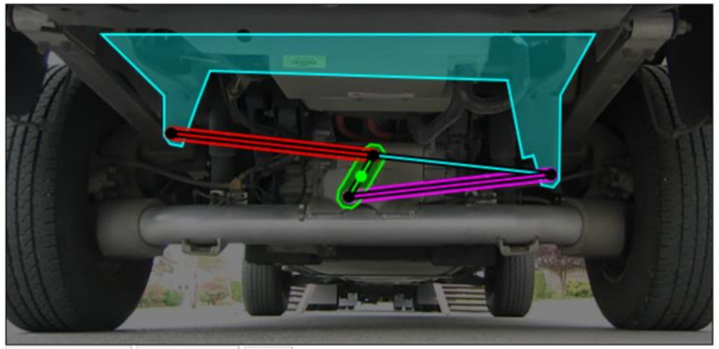
qFwoTCJDn6_vCx_UCFQAAAAAdAAAAABAD, http://dynref.engr.illinois.edu/aml.html

Four-bar linkage



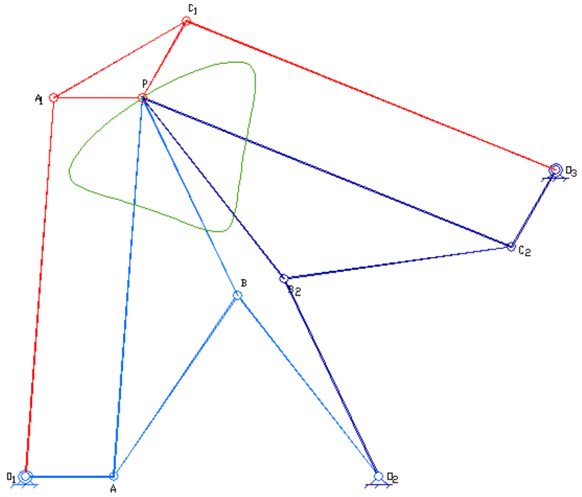
- can be used for many mechanical purposes, for example:
  - convert rotational motion to reciprocating motion. (e.g., pumpjack)
  - convert reciprocating motion to rotational motion (e.g., bicycle)
  - constrain motion (e.g., knee joint and suspension)
  - magnify force (e.g., parrotfish jaw)



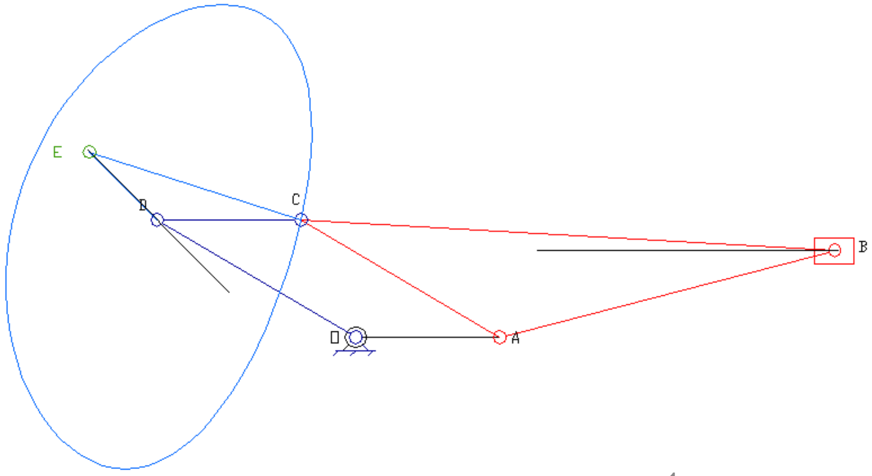


source: http://dynref.engr.illinois.edu/aml.html

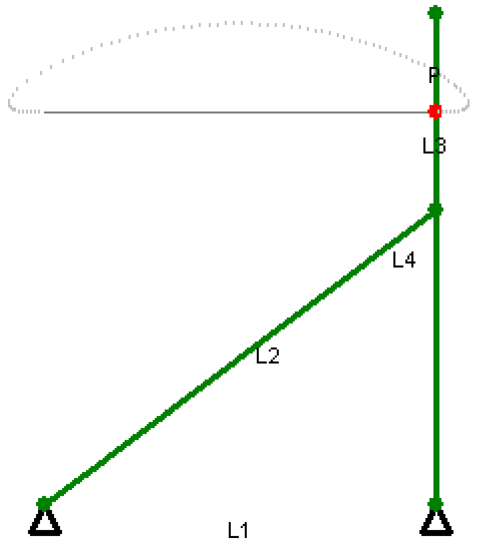
Cognate Linkage



- linkages that ensure the same input-output relationship or coupler curve geometry, while being dimensionally dissimilar.
- Roberts–Chebyschev Theorem:
  - each coupler curve can be generated by three different four-bar linkages.
- Overconstrained mechanisms can be obtained b connecting two or more cognate linkages together.



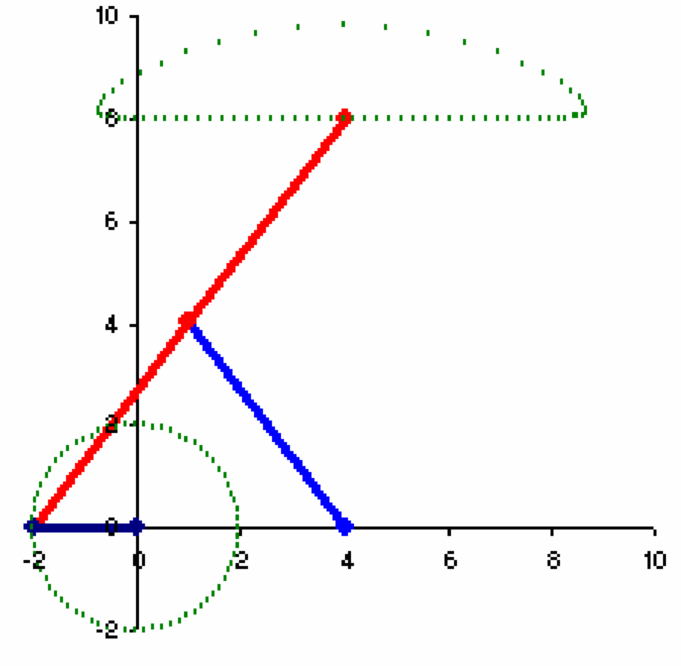
Chebyshev Linkage



- A mechanical linkage (4-bar) that converts rotational motion to approximate straight-line motion.
- invented by Pafnuty Chebyshev.
- Algebraic relation between the lengths:
  - $L_4 = L_3 + \sqrt{L_2^2 - L_1^2}$
- Lengths proportions:
  - L1:L2:L3 = 4:5:2
- From the proportions and constraints it follows that:
  - $L_2 = L_4$

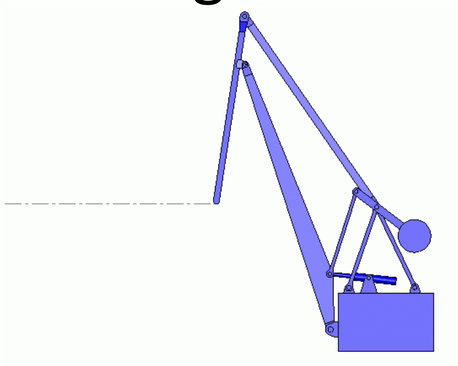
source: http://en.wikipedia.org/wiki/Chebyshev_linkage

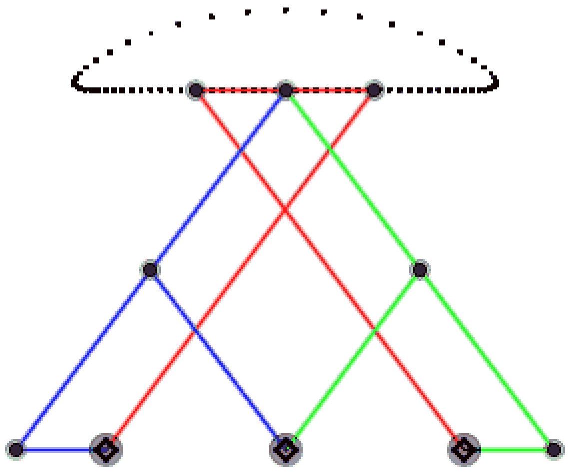
Chebyshev's Lambda Mechanism



- A four-bar mechanism that converts rotational motion to approximate straight-line motion with approximate constant velocity.

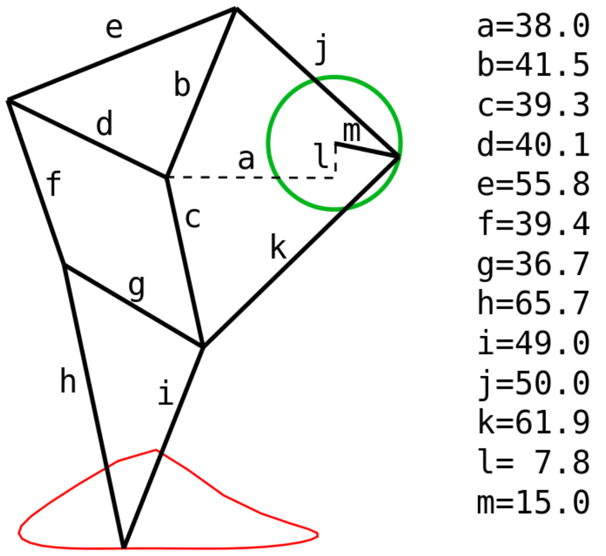
- Cognate linkage of the Chebyshev linkage



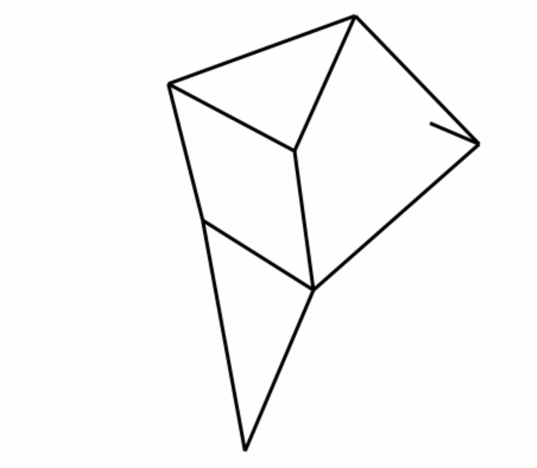


source: http://en.wikipedia.org/wiki/Chebyshev's_Lambda_Mechanism

Jansen Linkage

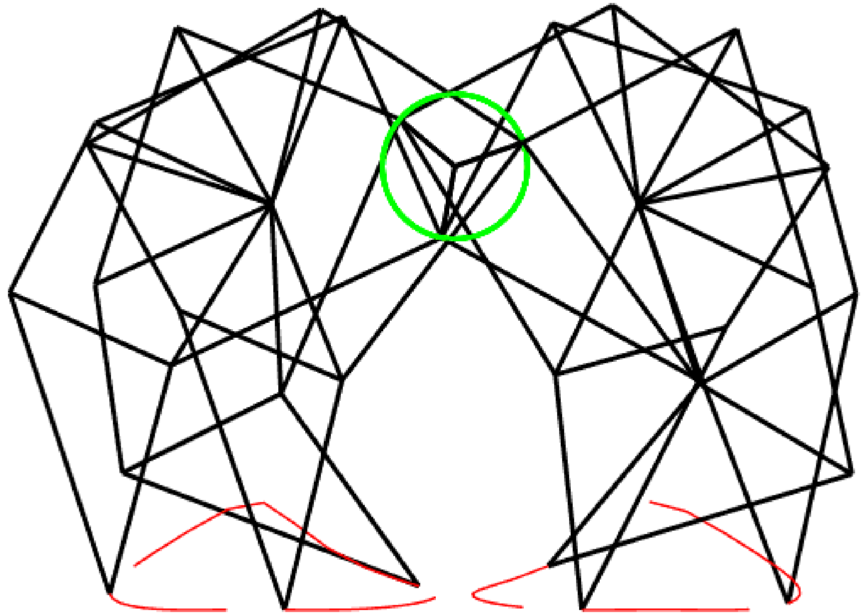


- A planar leg mechanism
- Designed by the kinetic sculptor Theo Jansen
- Generates a smooth walking motion.
- One degree of freedom
- Applications in mobile robotics and in gait analysis



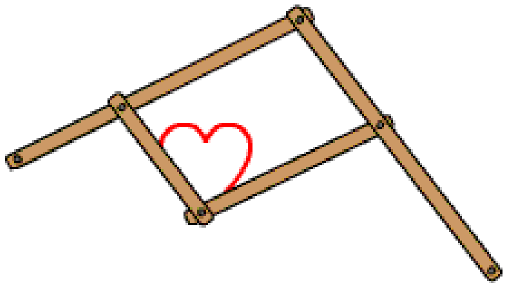
source: http://en.wikipedia.org/wiki/Jansen%27s_linkage

Jansen Linkage



source: http://en.wikipedia.org/wiki/Jansen%27s_linkage

Pantograph



- mechanical linkage connected in a manner based on parallelograms so that the movement of one pen, in tracing an image, produces identical movements in a second pen.
- If a line drawing is traced by the first point, an identical, enlarged, or miniaturized copy will be drawn by a pen fixed to the other.
- Using the same principle, different kinds of pantographs are used for other forms of duplication in areas such as sculpture, minting, engraving, and milling.

source:https://en.wikipedia.org/wiki/Pantograph

Major Research and development

“Geometric Constraint Solver using Multivariate Rational Spline Functions”

Gershon Elber, Myung-Soo Kim

The problem

- given n multivariate piecewise rational constraints

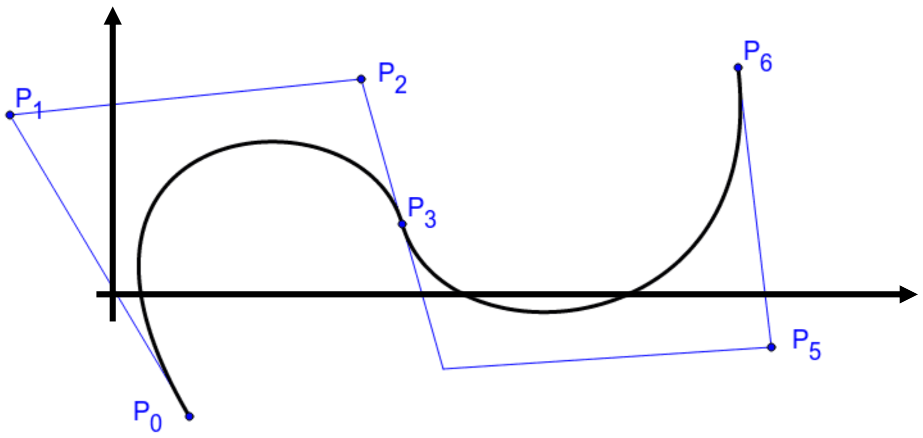
  $F_i(u_1, \dots u_{m-1}) = 0, \qquad i = 1, \dots, n$

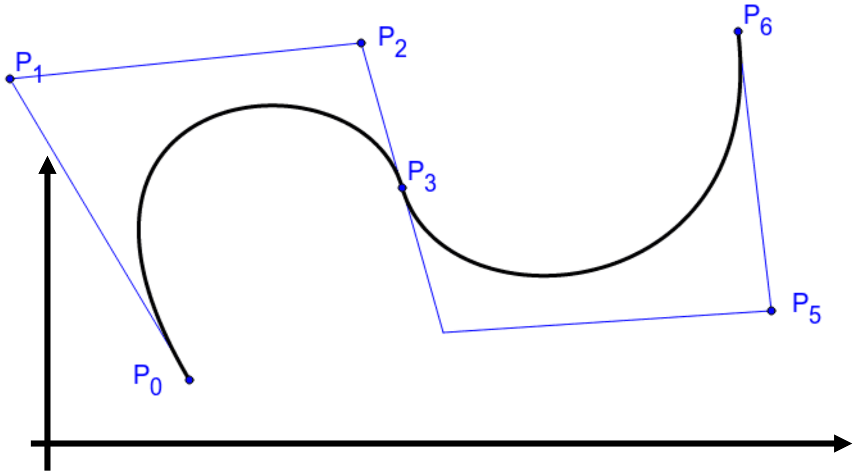
- We seek all $u^s \in \mathbb{R}^{m-1}, such\ that\ F_i(u^s) = 0\ for\ all\ i = 1, \dots, n$
- The $F_i's$ are represented as B-splines or Bezier multivariates scalar surfaces.
- Inequalities $F_i(u_1, \dots u_{m-1}) > 0$ are also supported.

source: “Geometric Constraint Solver using Multivariate Rational Spline Functions” Gershon Elber, Myung-Soo Kim

The convex-Hull test

- The convex-Hull property states that a Bézier curve is contained inside the convex hull of its control mesh.
- By the convex-hull property, the domain of $F_i(\mathrm{u})$ contains zeros only if the control coefficients of F_i have different signs.

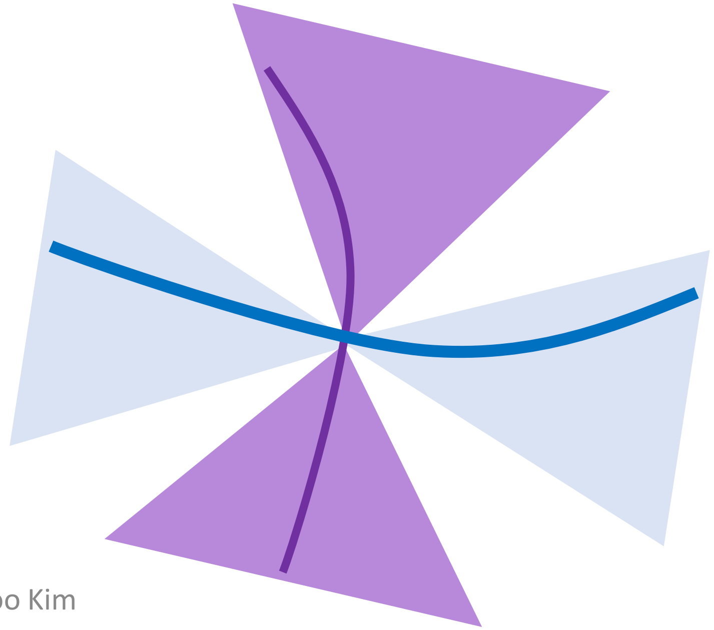




source: "Geometric Constraint Solver using Multivariate Rational Spline Functions" Gershon Elber, Myung-Soo Kim

The uniqueness test

- Bound the set of tangent directions in a pair of cones.
- If we position the pair of cones on any point on c, the entire curve will be contained in the cones.
- if the cones of tangent directions of $c0$ and $c1$ do not overlap (except for the apex), then $c0$ and $c1$ intersect at most once.
- Can be extended multivariates.



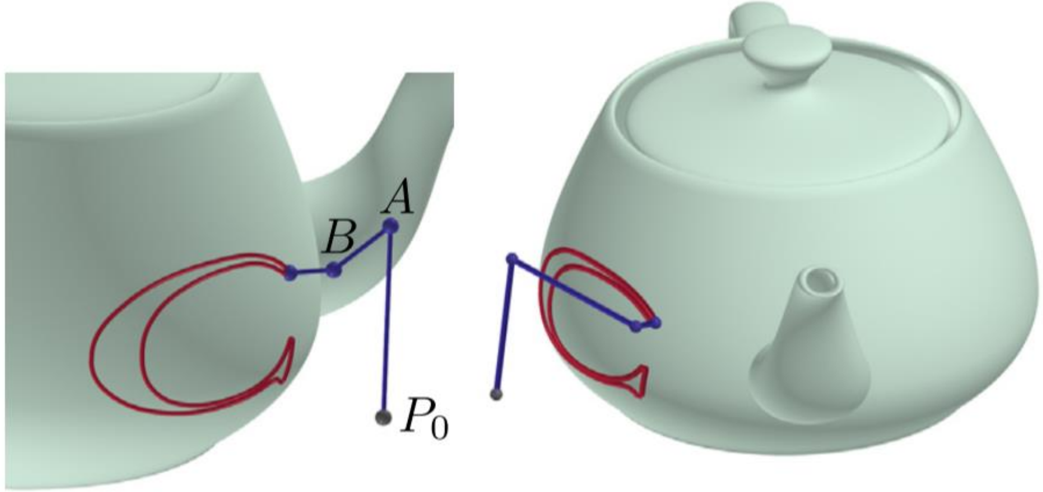
source: "Geometric Constraint Solver using Multivariate Rational Spline Functions" Gershon Elber, Myung-Soo Kim

The solver

- subdivision based.
- For each sub-domain:
  - The convex-hull test is performed: if all the control points have the same sign, the solution is not in the sub-domain and the sub-domain is trimmed from the search domain.
  - The uniqueness test is performed: if there is a unique solution, the subdivision stops and a numerical procedure is operated. the segment is numerically traced up to a user defined accuracy.
  - Otherwise, the subdomain $D \in \mathbb{R}^n$ is recursively divided until a condition for the existence of a single univariate solution segment can be met.
- At the end it generates a set of discrete points which are the simultaneous zero-set of F_i.
- Can support inequality constraints by checking if the control points have the same signs as the constraint.

source: “Geometric Constraint Solver using Multivariate Rational Spline Functions” Gershon Elber, Myung-Soo Kim

“Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.



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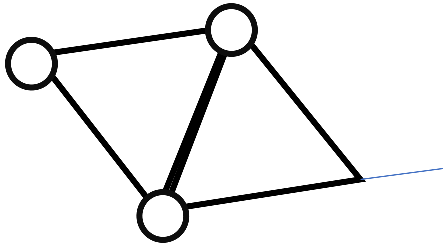
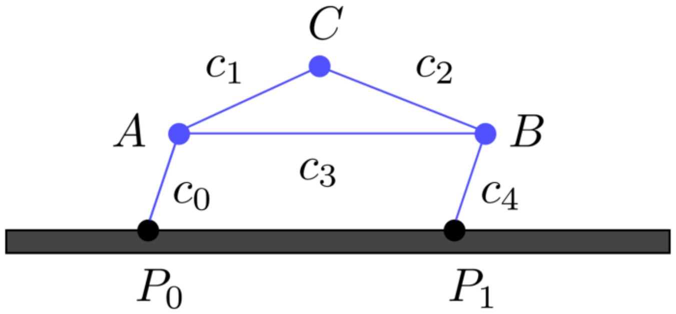
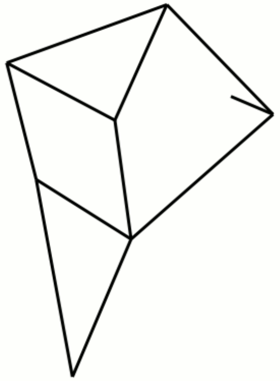
Motivation







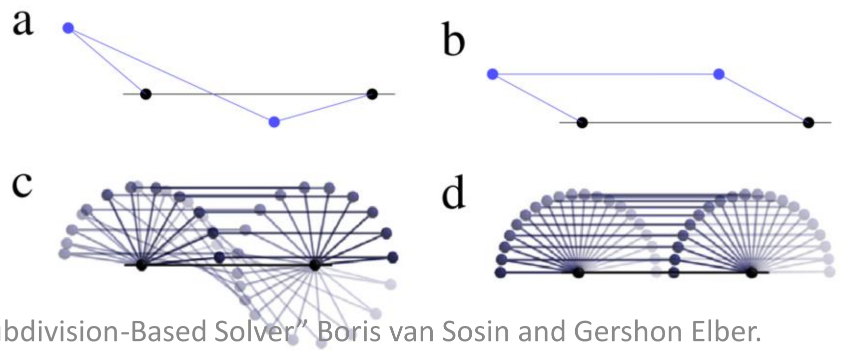
- Constraint systems are used in all computational geometry and CAD systems.
- In many applications, geometric constraint systems can get very complicated.
- Even a constraint as simple as a Euclidean distance between two points is quadratic
- Therefore, most applications require numerical solvers for constraints systems.
- The time it takes for most constraint systems solvers to run scales non-linearly, even exponentially, with the size of the problem.
- Therefore, decomposing the problem into a series of sub-problems in sequence can be very effective at speeding up the solution process.

source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

The problem

- Solving non-linear constraint systems by decomposing them into subsystems.
- Each sub-system is solved by using a subdivision-based polynomial solver.
- The input constraint system:
  - represented as Bezier or B-spline multivariate functions.
  - with DOF=0 or DOF=1
  - Contains equality and inequality constraints.

$$\begin{cases} |A - P_0|^2 - c_0^2 = 0, & c_0 = 0.9 \\ |A - B|^2 - c_2^2 = 0, & c_2 = 2 \\ |B - P_1|^2 - c_3^2 = 0, & c_3 = 0.9 \end{cases}$$



source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

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Algebraic representation of the constraints

- Constraints can be expressed in algebraic form as equations.
- Two types:
  - Zero constraints – have the form $f(x_1, x_2, \ldots, x_n) = 0$.
  - Inequality constraints- usually have the form $f(x_1, x_2, \ldots, x_n) \geq 0$
- The number of Zero constraints together with the number of variables determine the number of degrees of freedom of the system (DOF).
- Inequality constraints do not affect the total degrees of freedom of the system, but restrict the domain of the solution search.

source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

Algebraic representation of the constraints

- All the constraints are represented as B-spline functions.
- Extended to multivariate as a tensor product:
- $M(t) = \Sigma_{i_0=0}^{n_0} \Sigma_{i_0=0}^{n_0} \dots \Sigma_{i_0=0}^{n_0} \left(P_{i_0,i_1,\dots i_{k-1}} \prod_{j=0}^{k-1} B_{i_j}^{t_j}(t_j) \right)$
- Where:
  - $B_{i_j}^{t_j}$ - the B-spline basis functions of order q_j

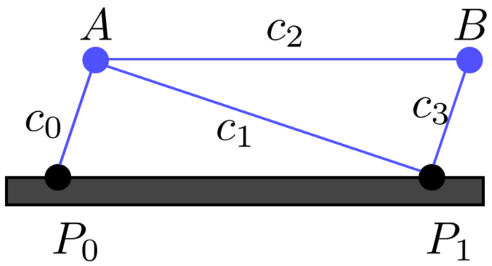
source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

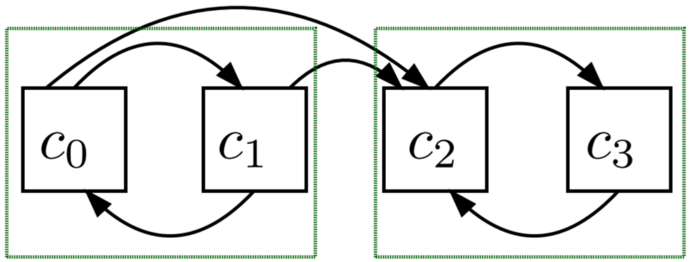
Constraint system

- A Constraint system is $M = M^{zero} \cup M^{ineq}$
- where:
  - $M^{zero} = \begin{cases} M_0^{zero}(\boldsymbol{t}) = 0 \\ M_1^{zero}(\boldsymbol{t}) = 0 \\ \vdots \\ M_{m-1}^{zero}(\boldsymbol{t}) = 0 \end{cases}, M^{ineq} = \begin{cases} M_0^{ineq}(\boldsymbol{t}) \geq 0 \\ M_1^{ineq}(\boldsymbol{t}) \geq 0 \\ \vdots \\ M_{p-1}^{ineq}(\boldsymbol{t}) \geq 0 \end{cases}$
  - $\boldsymbol{t} = (t_0, t_1, \dots t_{k-1})$ the variables
  - k is number of variables
  - $m > 0$ *number of zero constraints*.
  - $p \geq 0$ *number of inequality constraints*.

source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

Solution plan graph



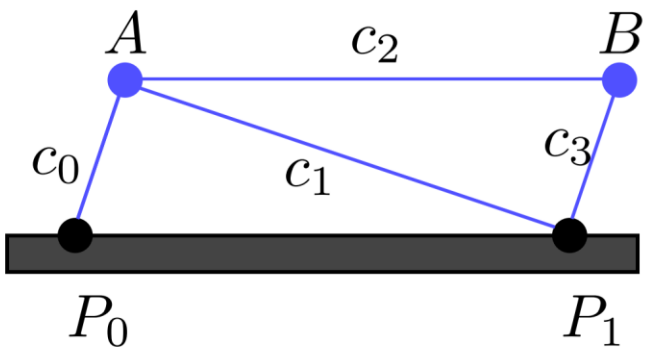


- Let M be a constraint system with m zero constraints and k variables.
- A solution plan graph is a graph $H_{plan} = (V_{plan}, E_{plan})$ with the following properties:
  - H_{plan} is a DAG (directed acyclic graph)
  - Each vertex $v_i \in V_{plan}$ represents a step in the solution plan with attached $constraints(v_i) = \{M_j\}_{v_i}$, the set of constraints of the subproblem solved in this step.
  - For $v_i \neq v_j$, $constraints(v_i) \cap constraints(v_j) = \phi$ and $\bigcup_{v \in V_{plan}} constraints(v_i) = M$.
  - There is an edge $v_i \rightarrow v_j \in E_{plan}$ if the subproblem v_j is dependent on v_i.

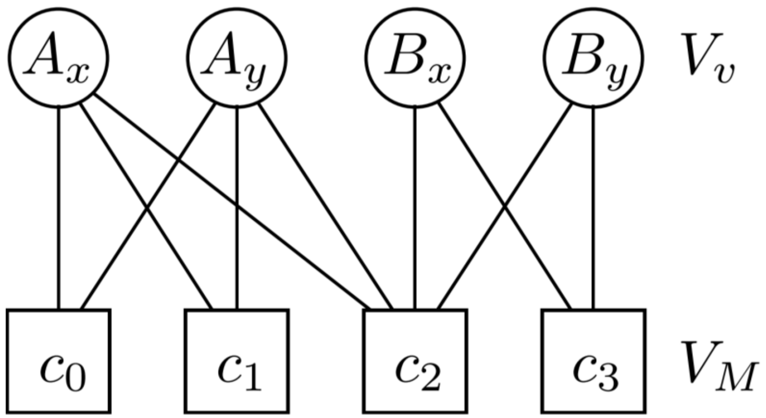
source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

Variable-constraint graph

- A bipartite graph $G = (V_v \cup V_M, E)$ with the following properties:
  1. $V_v = \{V_{t_i}\}_{i=0}^{k-1}$, a vertex for each variable t_i in the vector $\boldsymbol{t}$
  2. $V_M = \{v_{M_j}\}_{j=0}^{m-1}$, a vertex for each constraint in M^{zero}.
  3. There is an edge $(v_t, v_{M_j}) \in E$ iff constraint M_j is dependent on variable t_i.



$$M^{zero} = \begin{cases} |A - P_0|^2 - c_0^2 = 0, & c_0 = 0.9 \\ |A - P_1|^2 - c_1^2 = 0, & c_1 = 2.5 \\ |A - B|^2 - c_2^2 = 0, & c_2 = 2 \\ |B - P_1|^2 - c_3^2 = 0, & c_3 = 0.9 \end{cases}$$



source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

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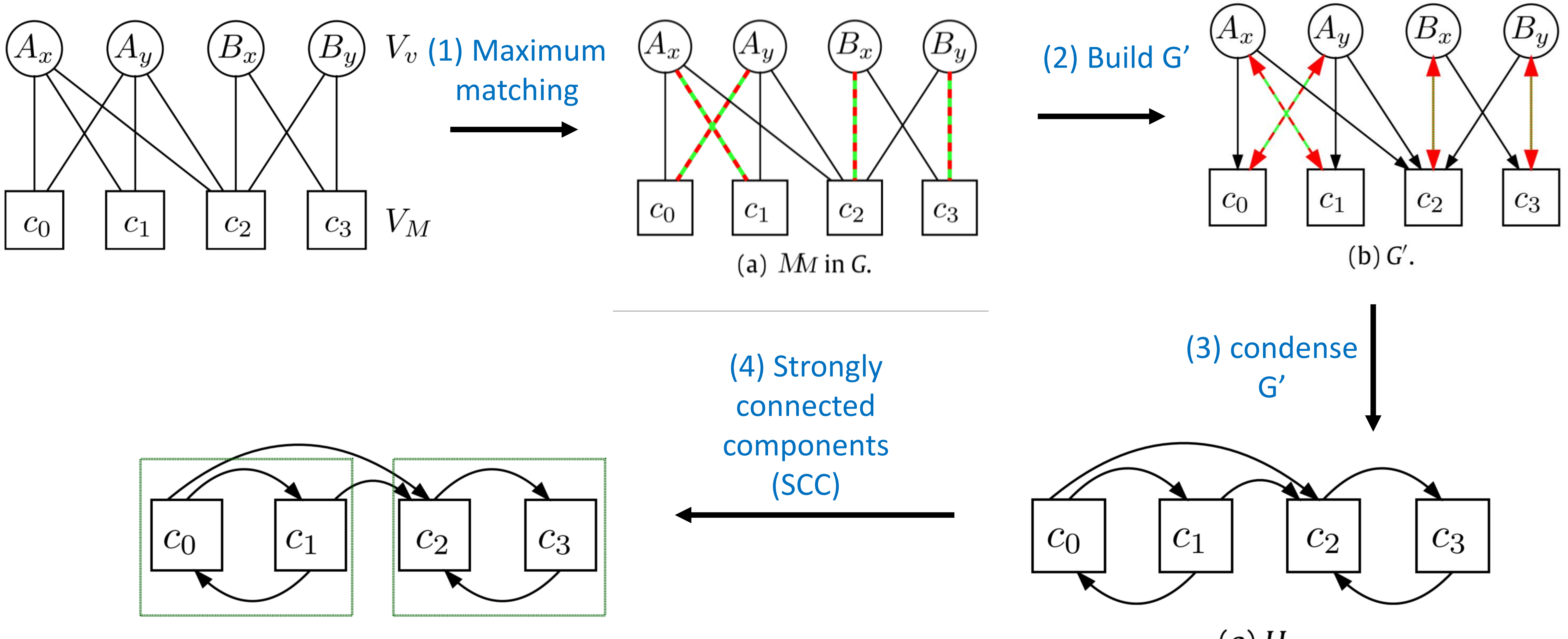
Building of H_{plan} from G



(a) M in G.

(b) G'.

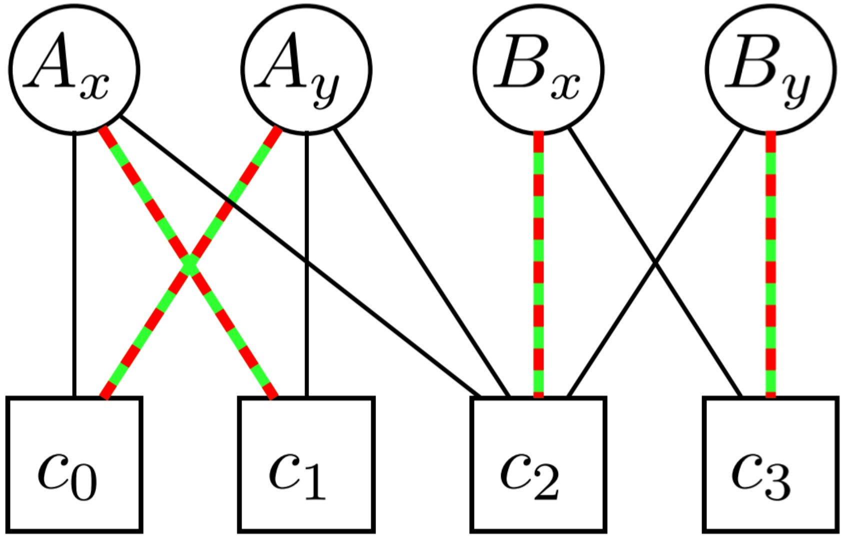
(c) H.

(d) SCCs of H.

source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

Building the solution graph H_{plan} from the variable-constraints graph $G = (V_v \cup V_M, E)$

1. Finding a maximum matching, M_m in G

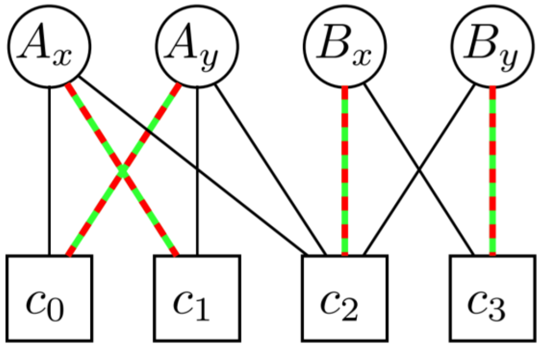


(a) M in G.

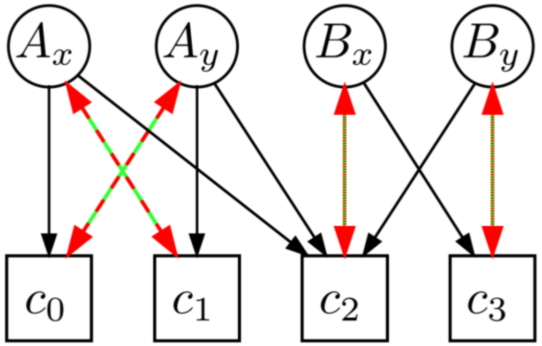
Building the solution graph H_{plan} from the variable-constraints graph $G = (V_v \cup V_M, E)$

2. Building a new directed graph $G' = (V', E')$ where:

- $V' = V_v \cup V_M$
- converting each of the matched edges into a pair of anti parallel directed edges.
- All the unmatched edges are copied to G’ as directed edges from the variable vertices to the constraint vertices.



(a) M in G.



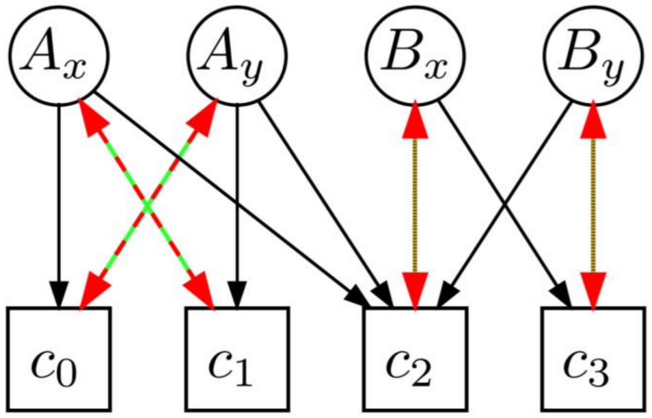
(b) G'.

source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

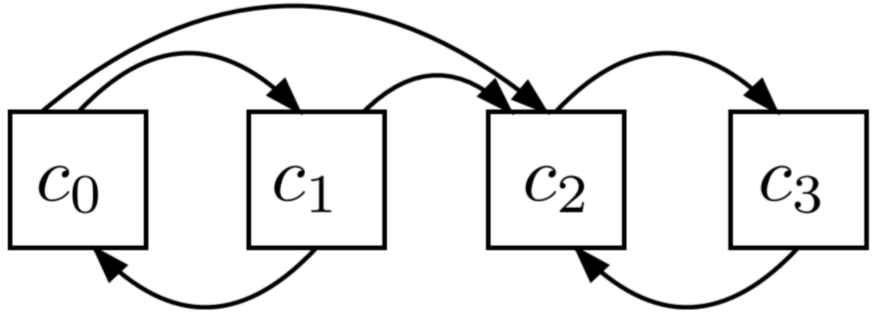
Building the solution graph H_{plan} from the variable-constraints graph $G = (V_v \cup V_M, E)$

3. Condensing G' to build H

by taking the set of vertices of G’ and connecting a pair of vertices v_{M_i}, v_{M_j} by a directed edge $v_{M_i} \rightarrow v_{M_j}$ iff in G’ there is a directed path from v_{M_i} to v_{M_j} going through a single variable vertex.



(b) G'.

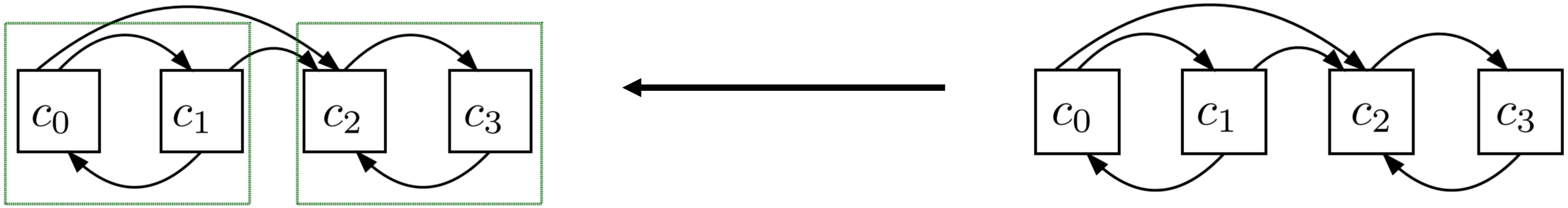


(c) H.

source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

Building the solution graph H_{plan} from the variable-constraints graph $G = (V_v \cup V_M, E)$

4. H_{plan} is the Strongly connected components (SCC) graph of H (without inequality constraints).



(d) SCCs of H.

(c) H.

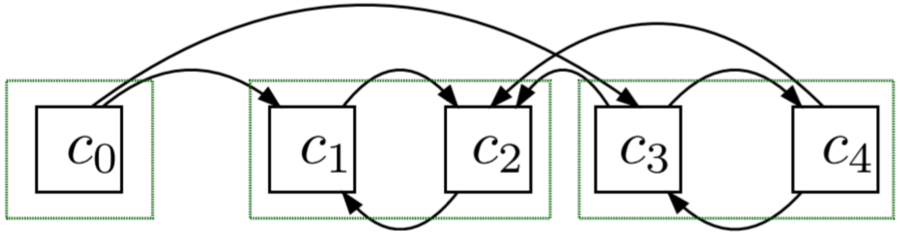
Building the solution graph H_{plan} from the variable-constraints graph $G = (V_v \cup V_M, E)$

5. Adding inequality constraints to H_{plan}

The inequality constraints are added to the subsystems in which they can be solved.

Each inequality constraint M_i^{ineq} is added to the subsystem in which at least one of the variables on which M_i^{ineq} depends is being solved for, in the subsystem, and all the variables which are not being solved for in the current subsystem, already have solutions.

The solution phase



- The subsystems in H_{plan} are solved in a topological order.
- Solving in topological order assuring that when a subsystem is solved, there are already values assigned to all the variables which are required to solve it.
- When the first subsystem is solved, the computed values are assigned to the variables in the subsystem.
- Zero dimensional or univariate assigned as a solution to a variable.
- Univariate solutions need to be parameterized in order to have the same representation as the constraints for further processing.
- The parametrization of univariate solutions is done either by parameterizing the piecewise-linear solutions directly, or by fitting a parametric curve (a B-spline curve) which approximates the piecewise-linear solution.

source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

Solving the subsystems (vertices in H_{plan})

- For each subsystem, all the variables that already have solutions are applied to the constraints.
- For zero dimensional solutions, the constraints multivariates are reduced to iso-parametric sub-multivariate.
- For univariate solutions, symbolic composition of the univariate solution B-spline into the constraint multivariates is performed.

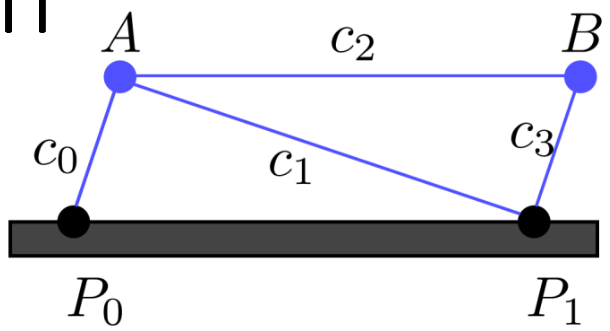
source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

Symbolic composition of the univariate solution

- If the variable vector of the problem is $\boldsymbol{t} = (t_0, t_1, \ldots, t_{k-1})$, and we have univariate solution for $t_0, \ldots t_{l-1}$, parameterized as $t_0(v), \ldots t_{l-1}(v)$, the constraint M_i undergoes the composition: $M_i(\tau(v), t_1, \ldots, t_{k-1}) = M_{i,comp}$

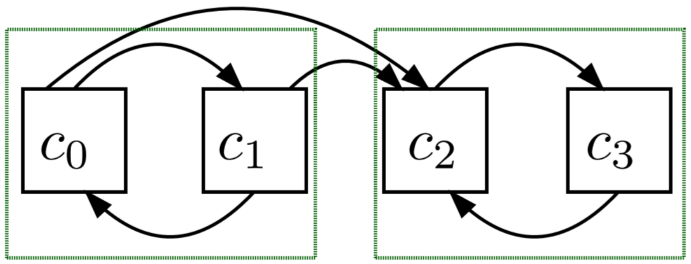
Results and examples

Example – 2D point-and-bar problem



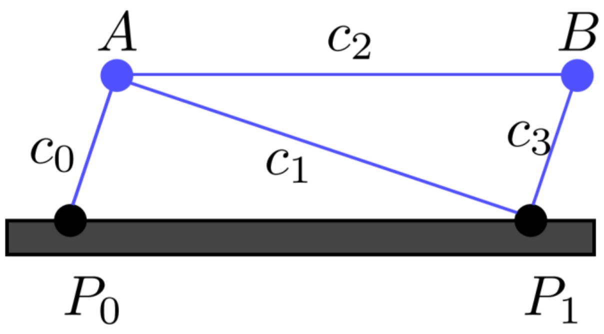
- Well-constrained system
- Algebraic representation:

  - $$M^{zero} = \begin{cases} |A - P_0|^2 - c_0^2 = 0, & c_0 = 0.9 \\ |A - P_1|^2 - c_1^2 = 0, & c_1 = 2.5 \\ |A - B|^2 - c_2^2 = 0, & c_2 = 2 \\ |B - P_1|^2 - c_3^2 = 0, & c_3 = 0.9 \end{cases}$$



(d) SCCs of H.

2D point-and-bar problem



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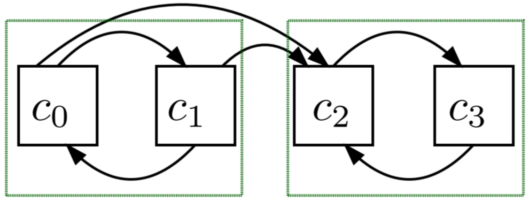
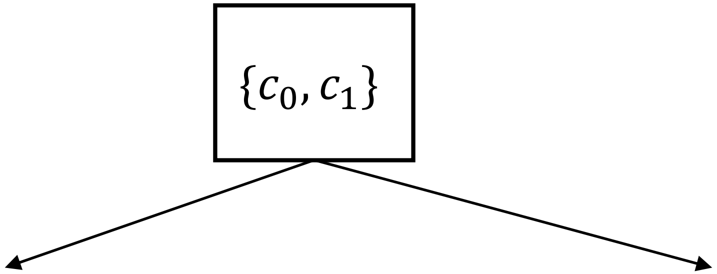


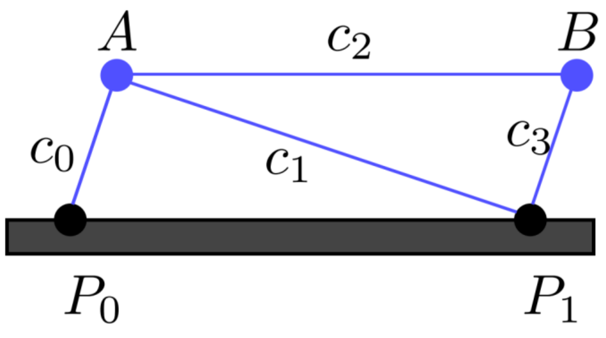
(d) SCCs of H.

- First the subsystem $\{c_0, c_1\}$ is solved and finds A_x, A_y



$(A_x = -1.36, A_y = 0.825)$ $(A_x = -1.36, A_y = 0.825)$

2D point-and-bar problem



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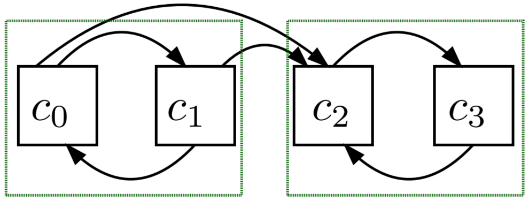

(d) SCCs of H.

- Next, the subsystem $\{c_2, c_3\}$ is solved and finds B_x, B_y

$\{c_0, c_1\}$

$(A_x = -1.36, A_y = 0.825)$

$(A_x = -1.36, A_y = -0.825)$

$(A_x, A_y, B_x, B_y) = (-1.36, 0.825, 0.205, -0.421)$

$(A_x, A_y, B_x, B_y) = (-1.36, 0.825, 0.64, 0.825)$

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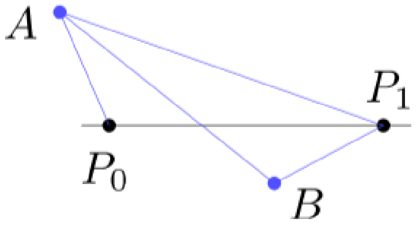
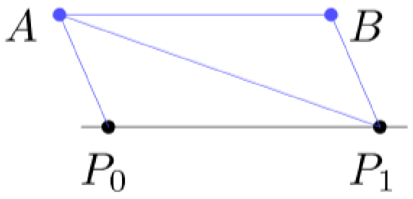
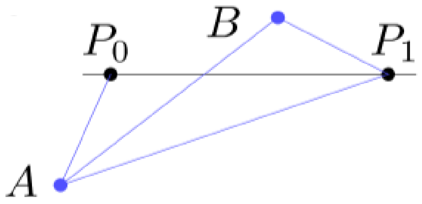
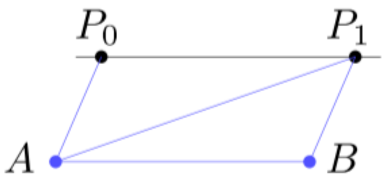


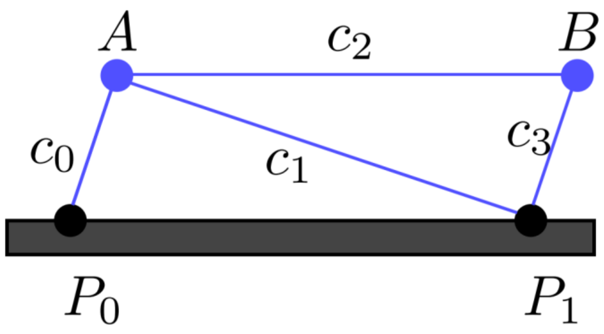






source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

2D point-and-bar problem



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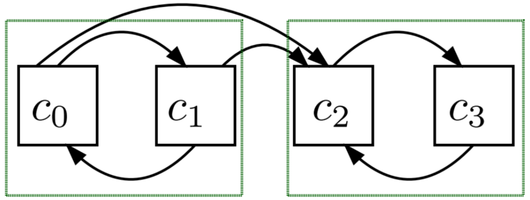

(d) SCCs of H.

- Next, the subsystem $\{c_2, c_3\}$ is solved and finds B_x, B_y

$\{c_0, c_1\}$

$(A_x = -1.36, A_y = 0.825)$ $\{c_2, c_3\}$

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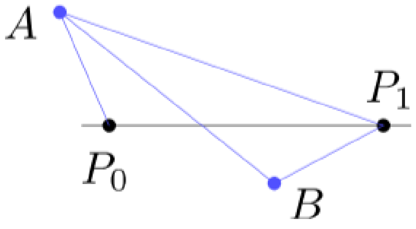
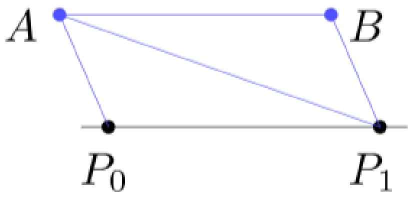
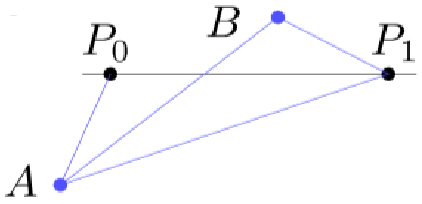
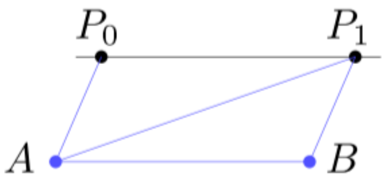


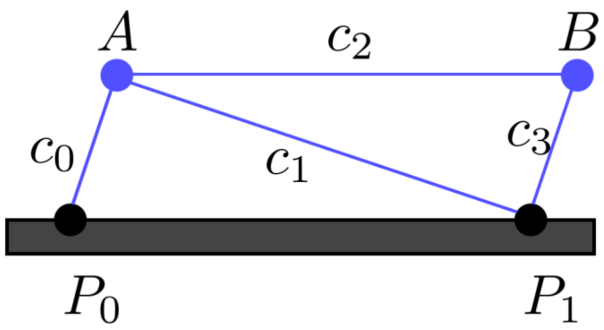






source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

2D point-and-bar problem



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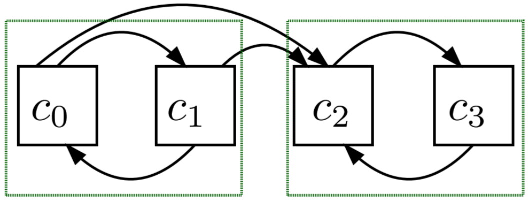

(d) SCCs of H.

- Adding inequality constraints to the system:

- $M^{ineq} = \begin{cases} A_y \geq 0 \\ z\big((B - p_1) \times (A - P_1)\big) \geq 0 \end{cases}$

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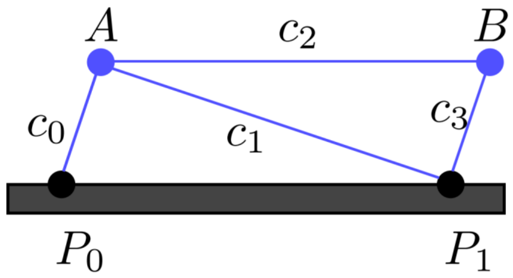
2D point-and-bar problem



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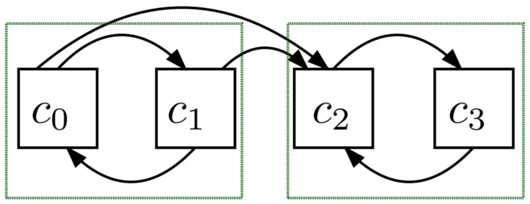
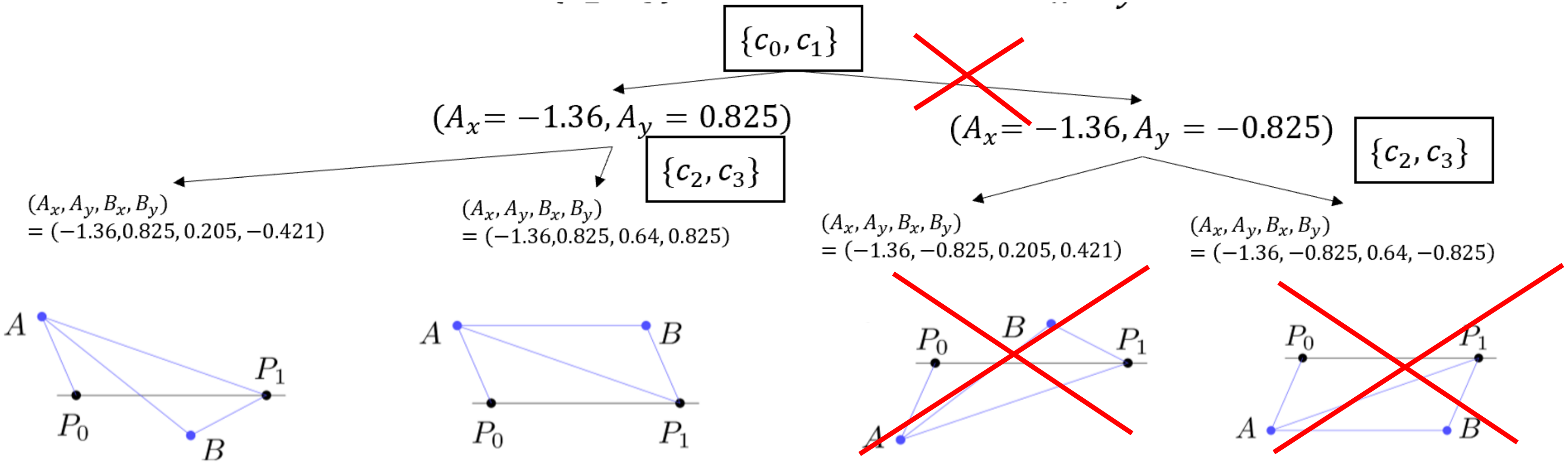
$$M^{ineq} = \begin{cases} A_y \geq 0 \\ z((B - p_1) \times (A - P_1)) \geq 0 \end{cases}$$



(d) SCCs of H.

- The constraint$z((B - p_1) \times (A - P_1)) \geq 0$ is added to the subsystem $\{c_2, c_3\}$



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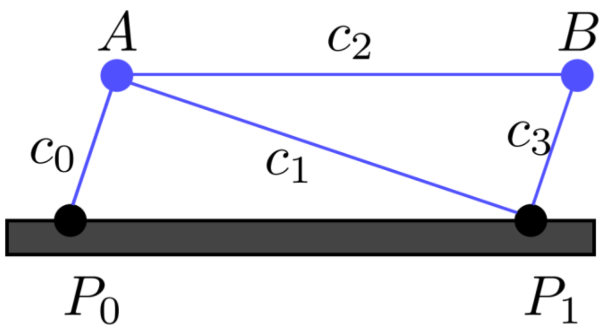
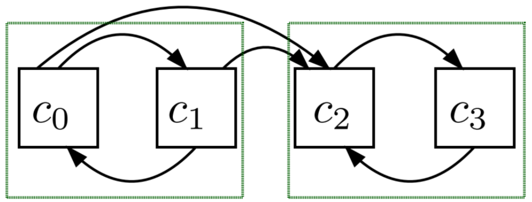
2D point-and-bar problem

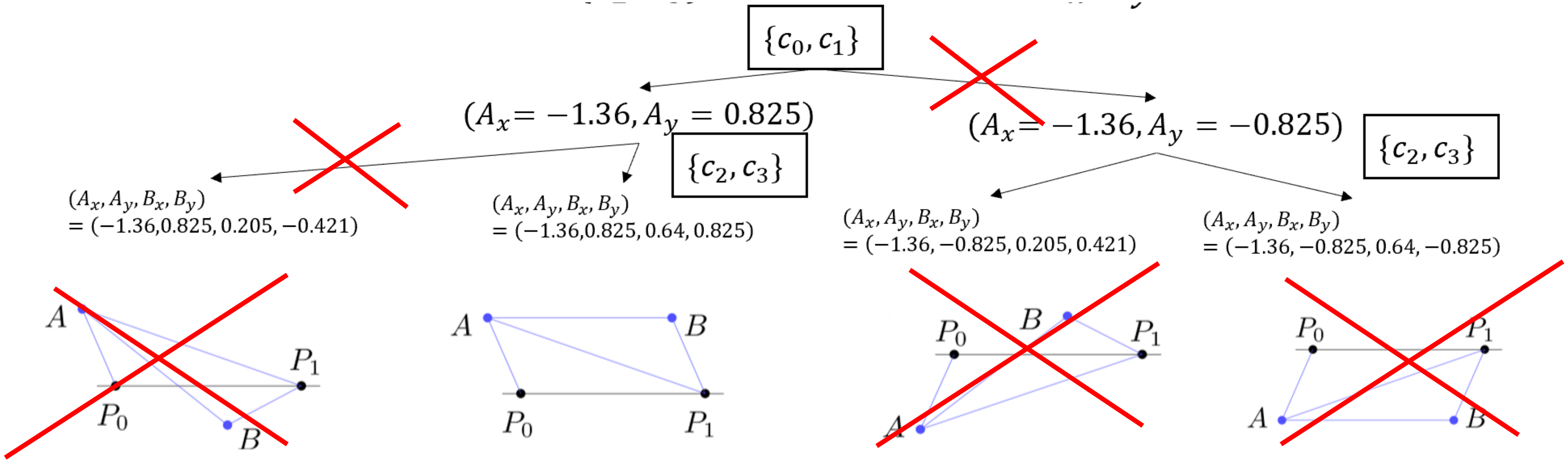


$$M^{zero} = \begin{cases} |A-P_0|^2 - c_0^2 = 0, & c_0 = 0.9 \\ |A-P_1|^2 - c_1^2 = 0, & c_1 = 2.5 \\ |A-B|^2 - c_2^2 = 0, & c_2 = 2 \\ |B-P_1|^2 - c_3^2 = 0, & c_3 = 0.9 \end{cases} \qquad M^{ineq} = \begin{cases} A_y \geq 0 \\ z((B-p_1)\times(A-P_1)) \geq 0 \end{cases}$$



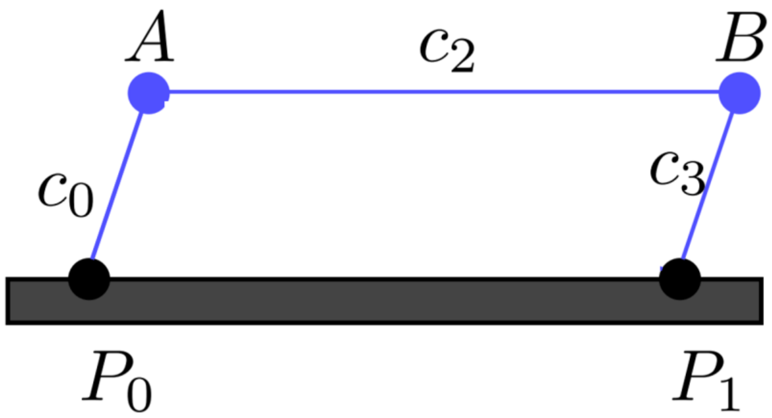
(d) SCCs of H.

- The constraint $A_y \geq 0$ is added to the system $\{c_0, c_1\}$



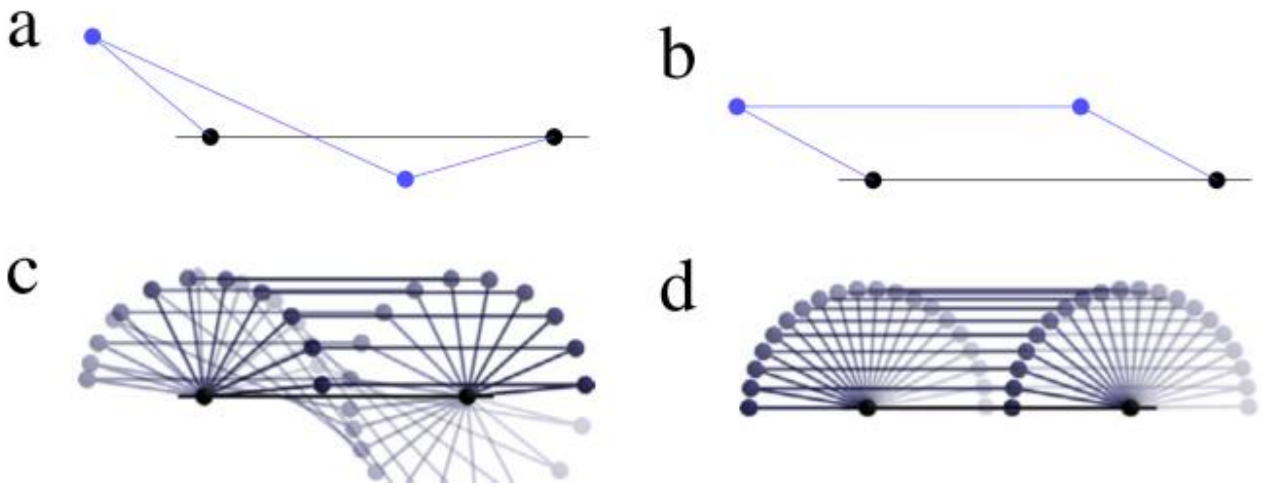
source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

4-bar linkage problem



- under-constrained system
- same inequality constraints:

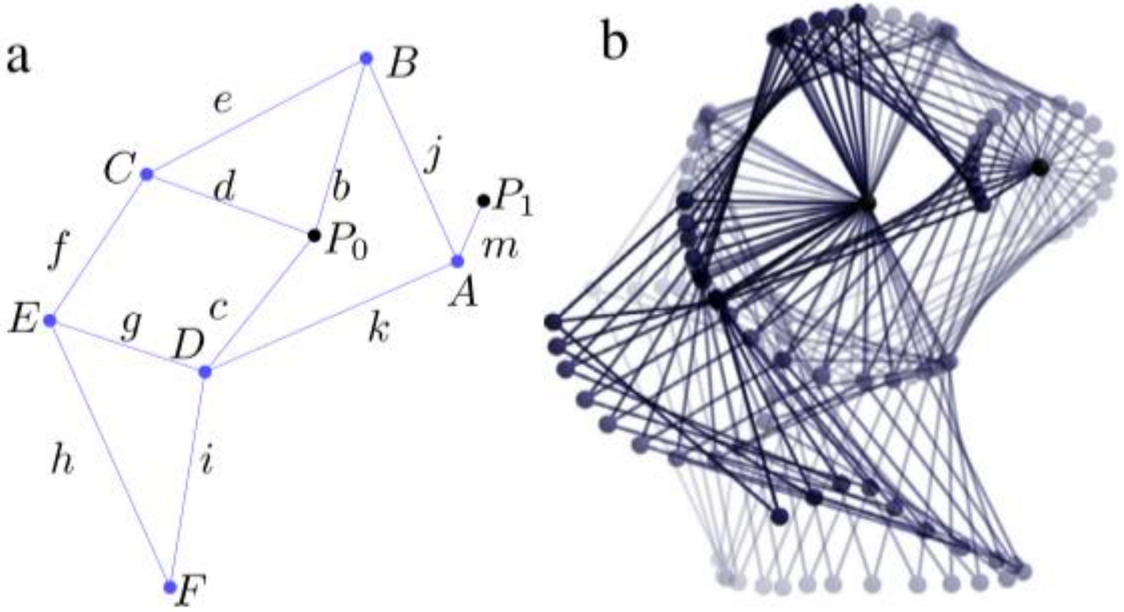
$$M^{ineq} = \begin{cases} A_y \geq 0 \\ z\big((B - p_1) \times (A - P_1)\big) \geq 0 \end{cases}$$



source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

The Jansen's linkage problem

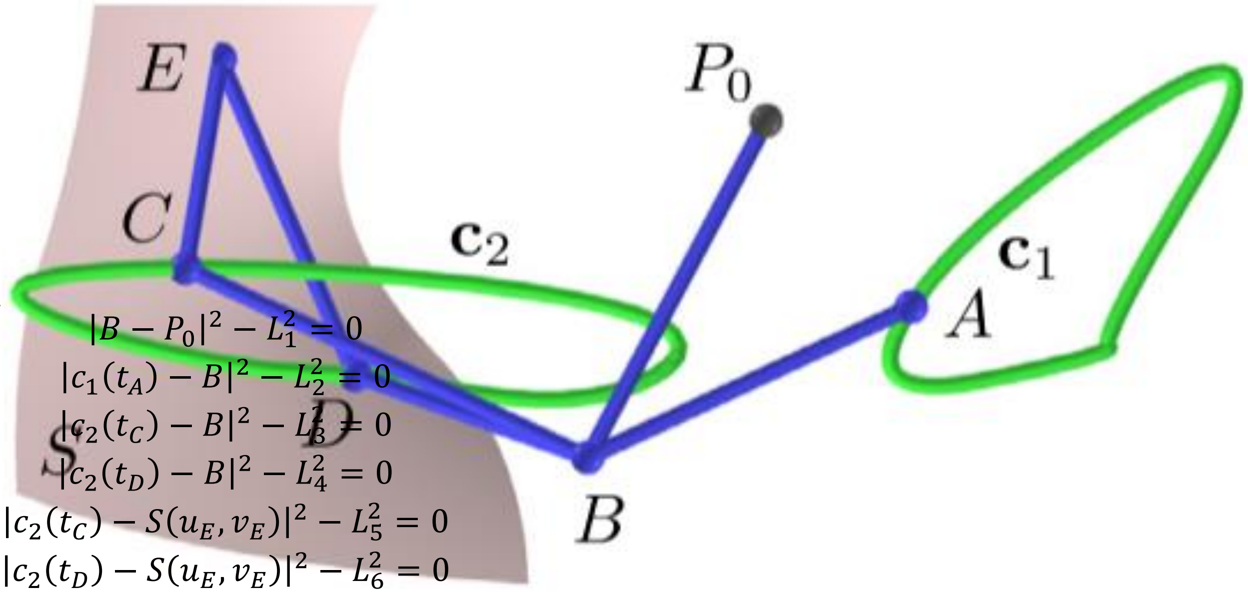
- Decomposed into 6 subsystems.
- Found 38 solutions without inequalities.
- Added inequalities:
- $$\begin{cases} (P_0 - B) \times (A - B) \geq 0 \\ (A - D) \times (P_0 - D) \geq 0 \\ (C - B) \times (P_0 - B) \geq 0 \\ (E - D) \times (F - D) \geq 0 \\ (D - E) \times (C - E) \geq 0 \end{cases}$$



source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

The kinematic over splines problem

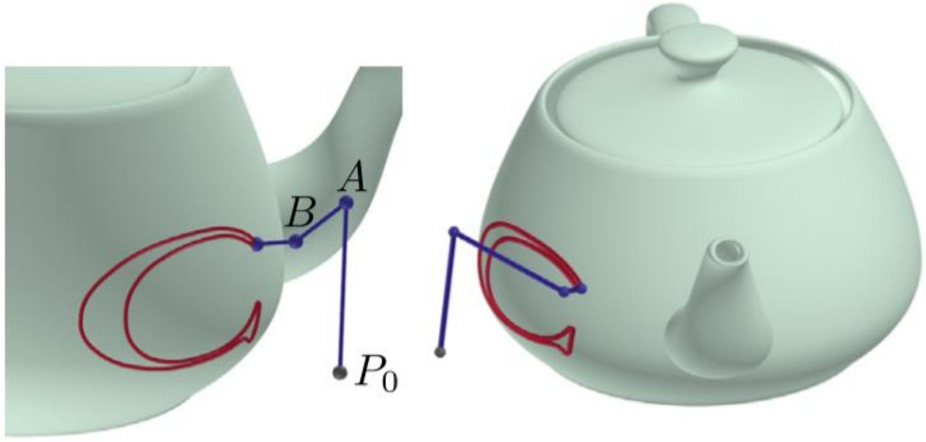
- Fixed point P_0, curves c_1, c_2, surface S.
- The variables:
  - $A = c_1(t_A)$
  - $B = (B_x, B_y)$
  - $C = c_2(t_c)$
  - $D = c_2(t_d)$
  - $E = S(u_E, v_E)$
- The constraints:



$$\begin{cases} |B - P_0|^2 - L_1^2 = 0 \\ |c_1(t_A) - B|^2 - L_2^2 = 0 \\ |c_2(t_C) - B|^2 - L_3^2 = 0 \\ |c_2(t_D) - B|^2 - L_4^2 = 0 \\ |c_2(t_C) - S(u_E, v_E)|^2 - L_5^2 = 0 \\ |c_2(t_D) - S(u_E, v_E)|^2 - L_6^2 = 0 \end{cases}$$

- Six constraints, 7 unknowns.
- Four disjoint solutions.

Inverse kinematics problem



- The tea pot is represented as $S(u, v)$
- The letter “C” is represented as a planar curve $c(t)$
- $c(t)$ is embedded in $S(u, v)$ by $c_s(t) = S\big(c(t)\big)$
- Constraints:
- $$\begin{cases} |c_s(t) - B|^2 = L_3^2 \\ \big\langle S_u\big(c(T)\big), B - c_s(t)\big\rangle = 0 \\ \big\langle S_v\big(c(T)\big), B - c_s(t)\big\rangle = 0 \\ |B - A|^2 = L_2^2 \\ |A - P_0|^2 = L_1^2 \\ z\big((B - A) \times (P_0 - A)\big) = 0 \end{cases}$$
- L_i are constants, $S_u = \frac{\partial S}{\partial u}, S_v = \frac{\partial S}{\partial v}$
- 6 constraints and 7 unknowns.
- Since the planar letter ‘‘C’’ is a quadratic curve, and the body of the teapot is a bi-cubic surface, the maximal polynomial orders of the first three constraints are 25, 23, and 23.

source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

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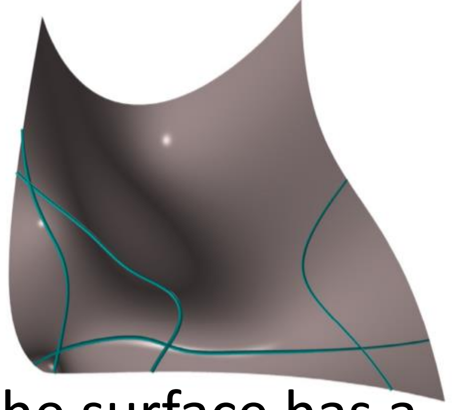
Flecnodal curves



- A flecnodal curve is defined as a locus of all the points at which the surface has a third-order contact with a ray.
- S(u,v) is a C^3 continuous surface.
- N(u,v) the unnormalized normal at u,v.
- The flecnodal curves of the surface are the solution to the following system:
- $$\begin{cases} \langle a^2 S_{uu}(u,v) + 2ab S_{uv}(u,v) + b^2 S_{vv}(u,v), n(u,v) \rangle = 0 \\ \langle a^3 S_{uuu}(u,v) + 3a^2 b S_{uuv}(u,v) + 3ab^2 S_{uvv}(u,v) + b^3 S_{vvv}(u,v), n(u,v) \rangle = 0 \\ a^2 + b^2 - 1 = 0 \end{cases}$$
- The constraint $a^2 + b^2 - 1 = 0$ was solved first.
- The solution propagated into the first two constraints allowing them to be solved more efficiently.

source: "Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver" Boris van Sosin and Gershon Elber.

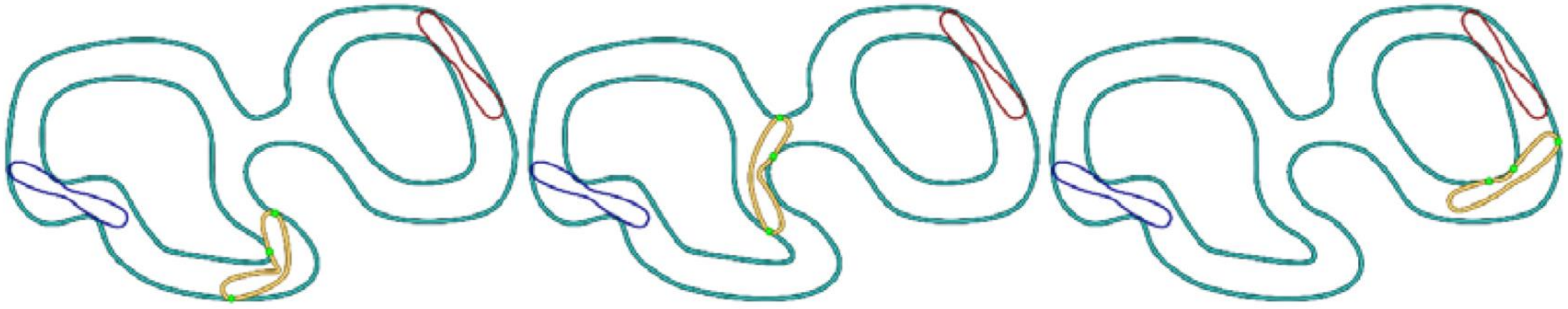
performance

Table 1
The performance of the framework with decomposition, compared to the subdivision-based solver without decomposition.

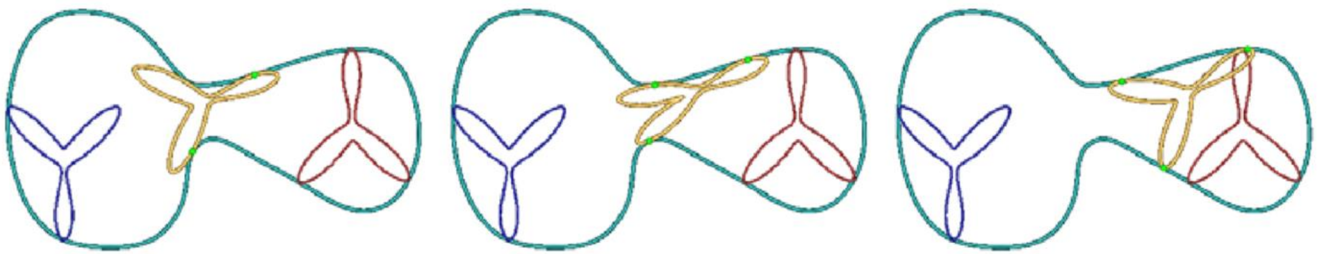
Problem	Time (seconds)		Speedup factor
	No decomposition	With decomposition	
2D point-and-bar	0.000474	0.000449	1.05
2D point-and-bar, with inequlity	0.00034	0.0003	1.13
Four-bar	0.27	0.21	1.32
Four-bar, with inequality	0.09	0.074	1.21
Jansen’s linkage	19730	61.2	322.3
Jansen’s linkage, with inequality	9690.4	2.7	3530.2
kinematic over splines	4544.2	7.6	599.3
Inverse kinematics	84.9	3.5	24.5
Flecnodal curves	0.70	0.36	1.96

source: “Solving Piecewise Polynomial Constraint Systems with Decomposition and a Subdivision-Based Solver” Boris van Sosin and Gershon Elber.

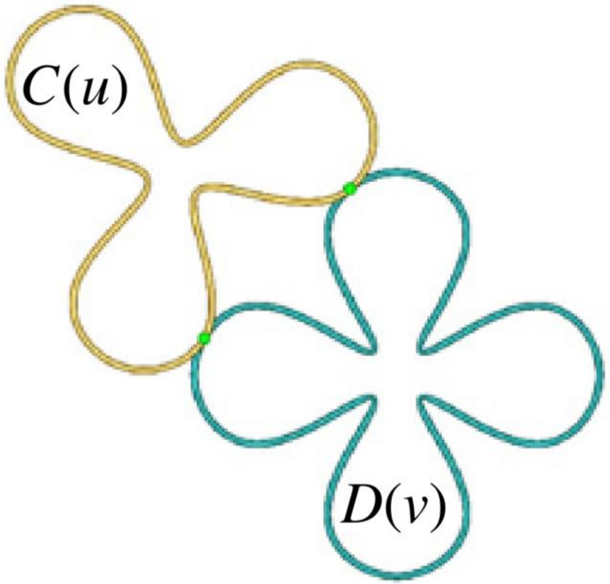
“Precise contact motion planning for deformable planar curved shapes” Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.



The problem



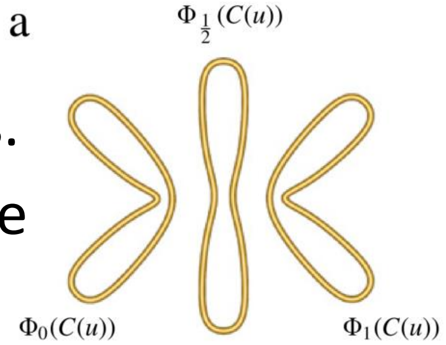
- Given a C^1 continuous robot $\phi_t(C(u))$ in parametric or implicit form
- And given parametric obstacles D(v), in the plane.
- ϕ_t is a one-parameter smooth freeform deformation of C(u)
- $\phi_t(C(u))$ has:
  - two translation DOF (x,y)
  - one rotation DOF θ
  - One DOF t provides shape control over ϕ_t
- Contact motion planning for the robot.



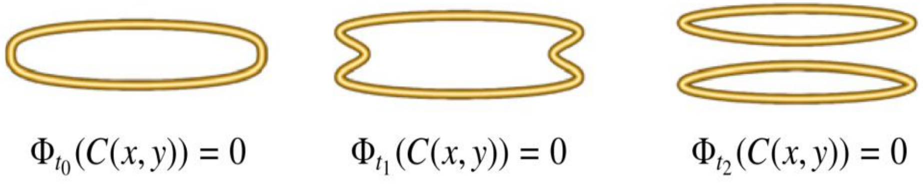
source: “Precise contact motion planning for deformable planar curved shapes” Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

Deformable robots representation

- Two ways of prescribing the robot (a planar deforming shape).
- Parametric:
  - $C_i(u_i), i = (1,2), u_i \in [0,1]$ two regular, smooth, parametric curves.
  - $\phi_t\big(C(u)\big) = (1-t)C_1(u) + tC_2(u)$ (Alternatively, $\phi_t(C(u))$ can be any $S(u,t)$)



- Implicit:
  - $C_i(x,y), = 0\ i = (1,2)$ two smooth implicit curves
  - $\phi_t\big(C(x,y)\big) = (1-t)C_1\big((x,y)\big) + tC_2((x,y))$ (Alternatively, can be any $v(x,y,t) = 0$)
  - The advantage is that it can change the topology



source: "Precise contact motion planning for deformable planar curved shapes" Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

Algebraic condition for k-contact motion

- $\left(\phi_t(C(u))_x, \phi_t(C(u))_y\right)$ C^1 continuous regular parametric curve.
- $D(v) = \left(D(v)_x, D(v)_y\right)$ stationary obstacle.
- Rigid transformation of $\phi_t(c(u))$: $T[\phi_t(c(u))] = R_\theta[\phi_t(c(u))] + (x, y)$
- The conditions for K-contact between $T[\phi_t(c(u_i))]$ and $D(v_i)$ i=1,…k:
  - $0 = R_\theta[\phi_t(c(u))]_x + x - D(v_i)_x$
  - $0 = R_\theta[\phi_t(c(u))]_y + y - D(v_i)_y$
  - $0 = F_i(u_i, v_i, \theta, t) = R_\theta[\phi_t(c(u))] \times D'(v_i)$

source: "Precise contact motion planning for deformable planar curved shapes" Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

Algebraic condition for k-contact motion

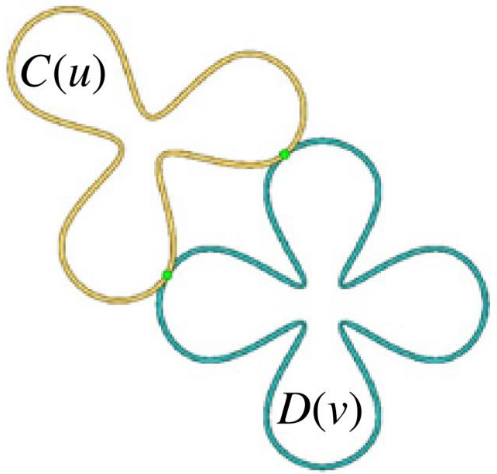
- Isolating x and y:
  - $\mathrm{x} = G_i(u_i, v_i, \theta, t) = D(v_i)_x - R_\theta\left[\phi_t\big(c(u)\big)\right]_x$
  - $\mathrm{y} = G_i(u_i, v_i, \theta, t) = D(v_i)_y - R_\theta\left[\phi_t\big(c(u)\big)\right]_y$

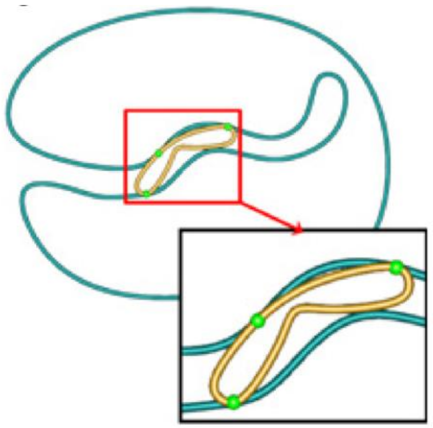
- We can write:
  - $0 = G_1(u_i, v_i, \theta, t) - G_i(u_i, v_i, \theta, t) \; for \;\; 2 \le i \le k$
  - $0 = H_1(u_i, v_i, \theta, t) - H_i(u_i, v_i, \theta, t) \; for \;\; 2 \le i \le k$
  - $0 = F_i(u_i, v_i, \theta, t) \;\; for \;\; 1 \le i \le k$
- Which is 3K-2 constraints in 2K+2 variables (u_i, v_i, θ, t)

Degrees of freedom

- For k=2 we have 4 constraints in 6 variables. If we fix t (i.e, rigid robot) we have 4 constraints and 5 variables. Therefore 1 degree of freedom.

- For k=3 we have 7 constraints in 8 variables. Therefore, 1 degree of freedom.





source: “Precise contact motion planning for deformable planar curved shapes” Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

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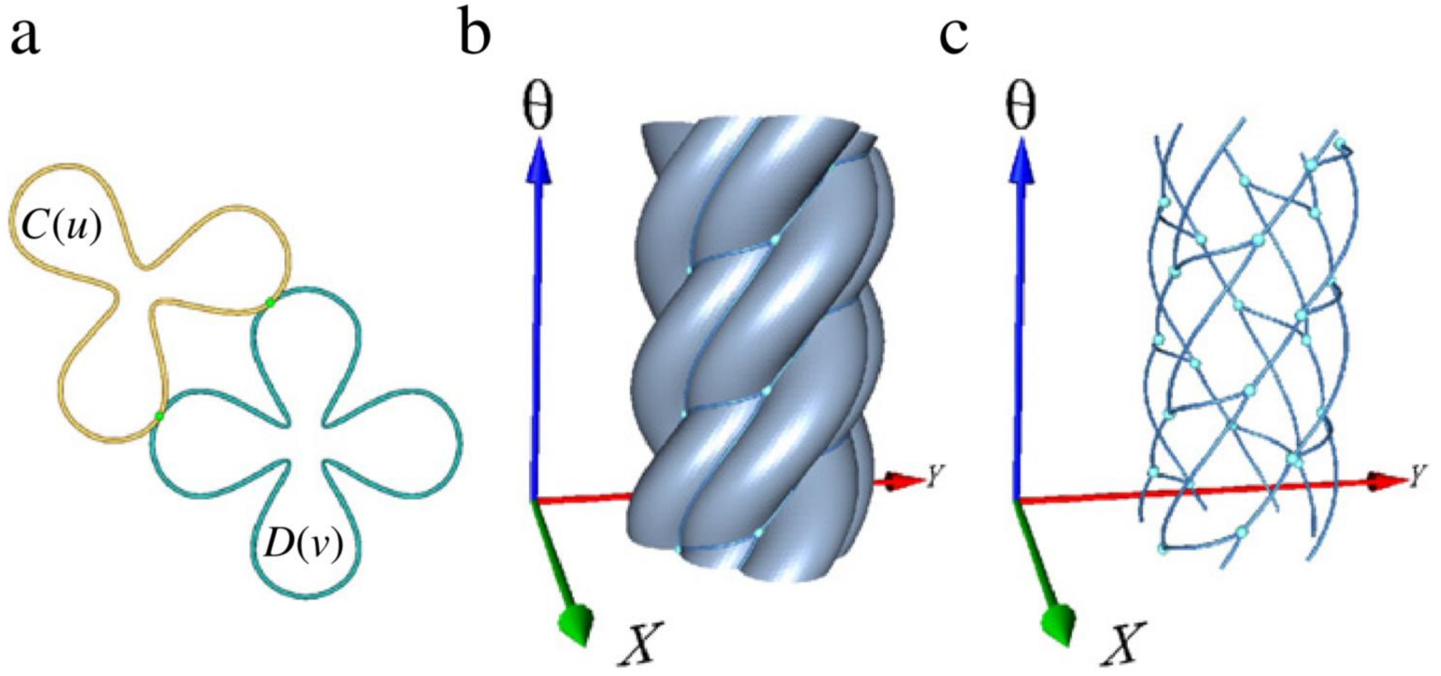
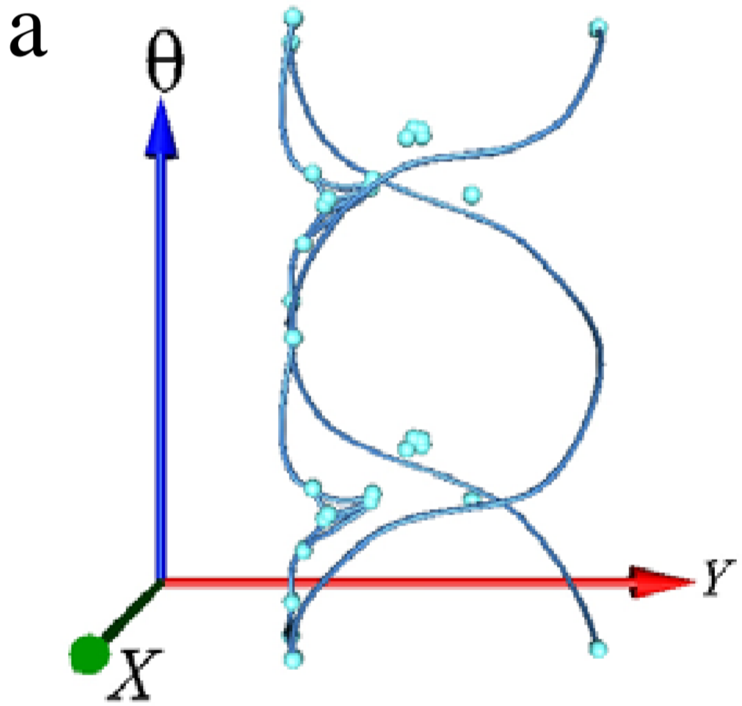
The K-contact motion graph

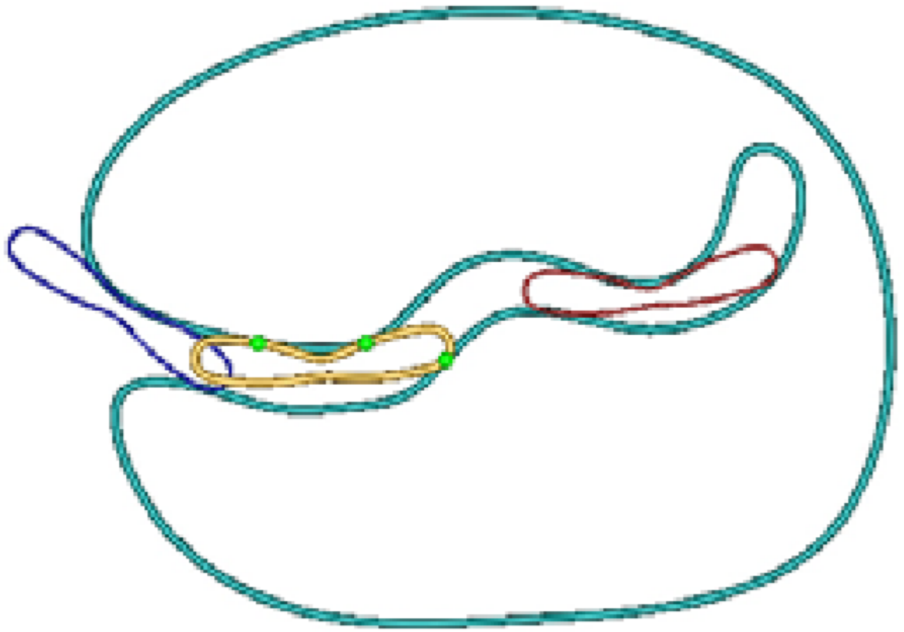
- Consists of 2 or 3 contact motions.
- Fixing $t = t^*$



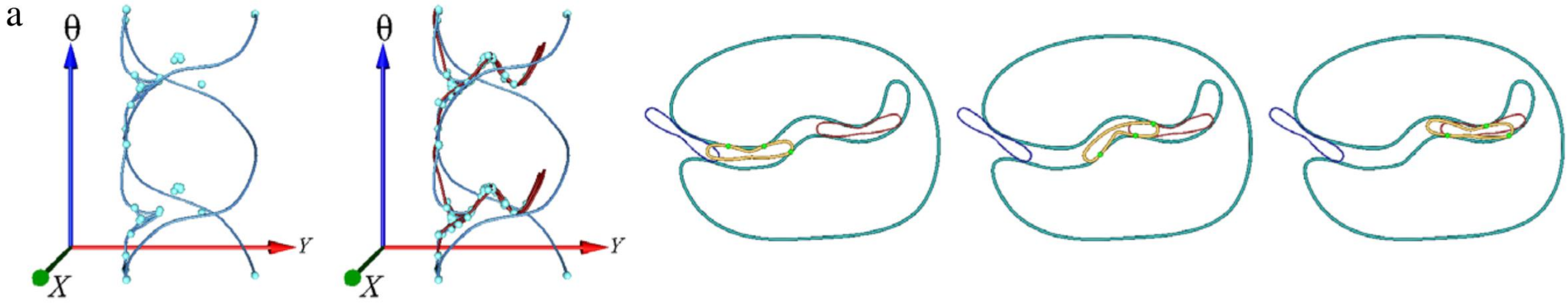
source: "Precise contact motion planning for deformable planar curved shapes" Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.

Disconnected components





3 – contact graph



source: “Precise contact motion planning for deformable planar curved shapes” Yong-Joon Kim, Gershon Elber, Myung-Soo Kim.