Volumetric Representations

Based on Cohen, Riesenfel and Elber book [1].

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27.01.2022

Trivariate Tensor Product

Definition:

Consider F, G and H, three sets of univaruate functions with intervals domains U, V and W, repescptively.

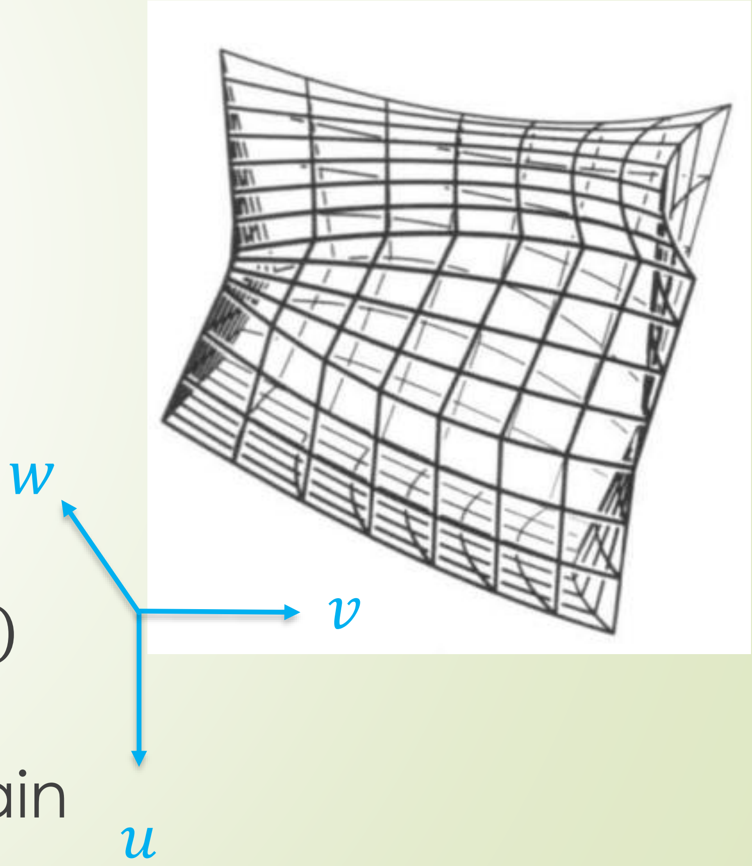
F = $\{f_i(u)\}_{i\,=\,0,\,m}$, G = $\{g_j(v)\}_{j\,=\,0,\,n}$ and

H = $\{h_k(w)\}_{k\,=\,0,\,l}$.

A volume formed by

$$T(u, v, w) = \sum_i \sum_j \sum_k P_{i,j,k} B_i(u) B_j(v) B_k(w)$$

is called a trivariate tensor product with domain $U \times V \times W$.



Iso-Parametric Surface/Curve

- The Iso-Parametric Surface is evaluated with a fixed value of w .

For $i = 0,..,m$ and $j = 0, \dots, n$, let $\gamma_{i,j} = \sum_k P_{i,j,k} B_k(\widetilde{w})$.

Then:

$$\mathrm{S}(\mathrm{u},\mathrm{v}) = T(u,v,\widetilde{w}) = \sum_i \sum_j \gamma_{i,j} B_i(u) B_j(v),$$

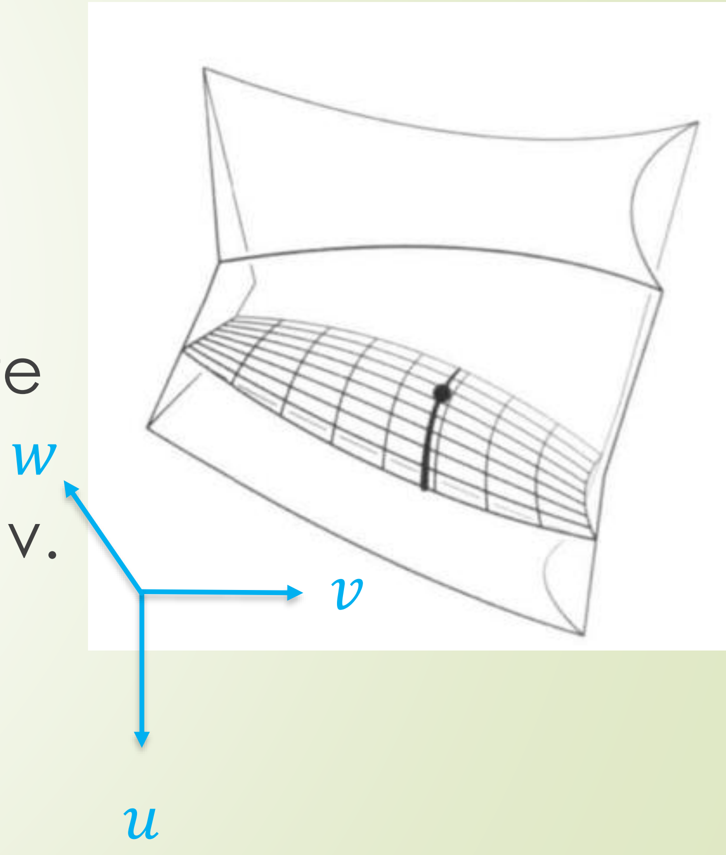
is an iso-parametric surface of T, which is just a bivariate tensor product surface.

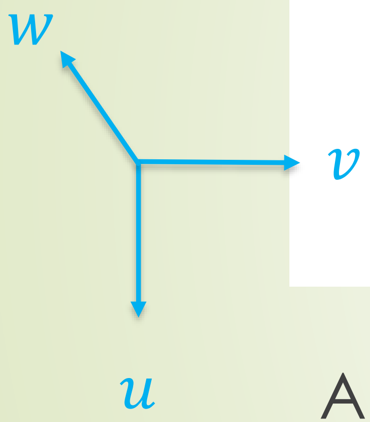
- The Iso-Parametric Curve with fixed values of w and v.

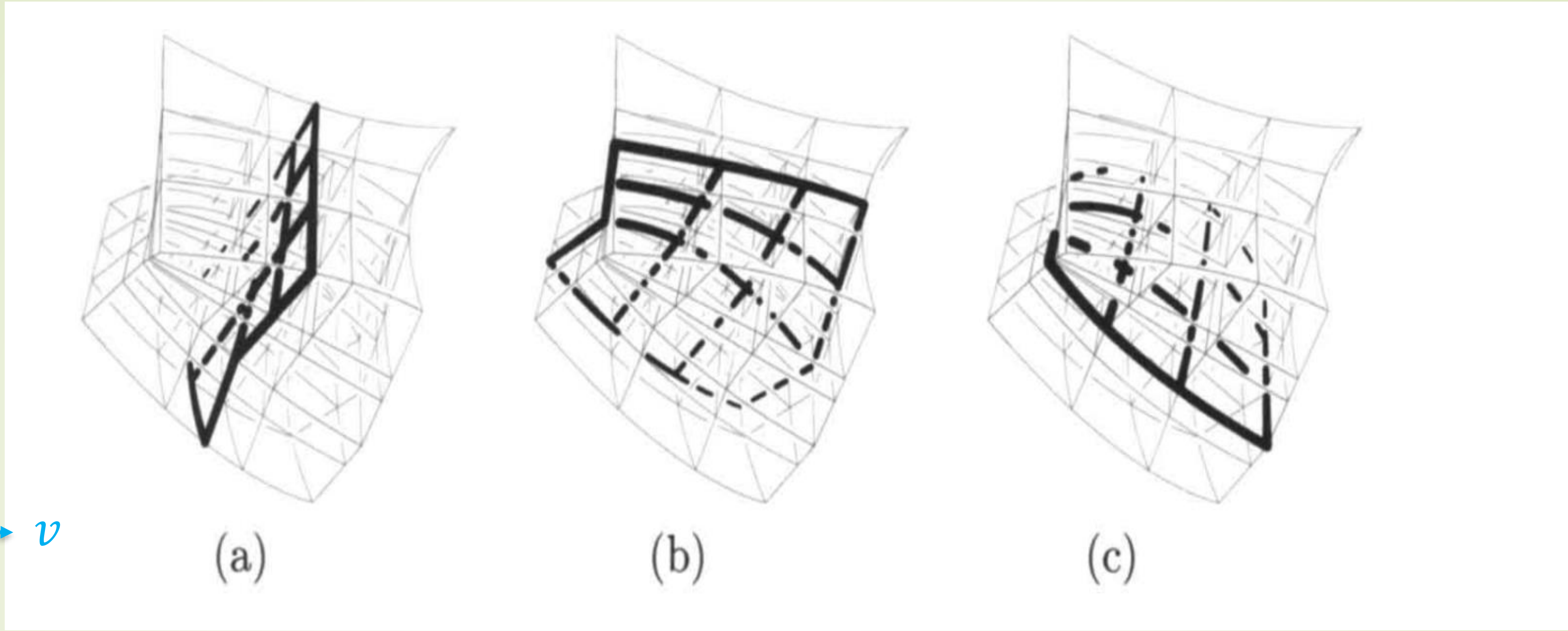
For $\mathrm{i} = 0,..,m$, let $\sigma_i = \sum_j \sum_k P_{i,j,k} B_k(\widetilde{w})\, B_j(\tilde{v})$.

Then:

$$C(u) = T(u,\tilde{v},\widetilde{w}) = \sum_i \sigma_i\, B_i(u).$$







A trivariate $T(u, v, w)$ with isoparametric surfaces:

a) $s(u, w) = T(u, v_0, w)$.

b) $s(v, w) = T(u_0, v, w)$.

c) $s(u, v) = T(u, v, w_0)$.

Partial Derivatives

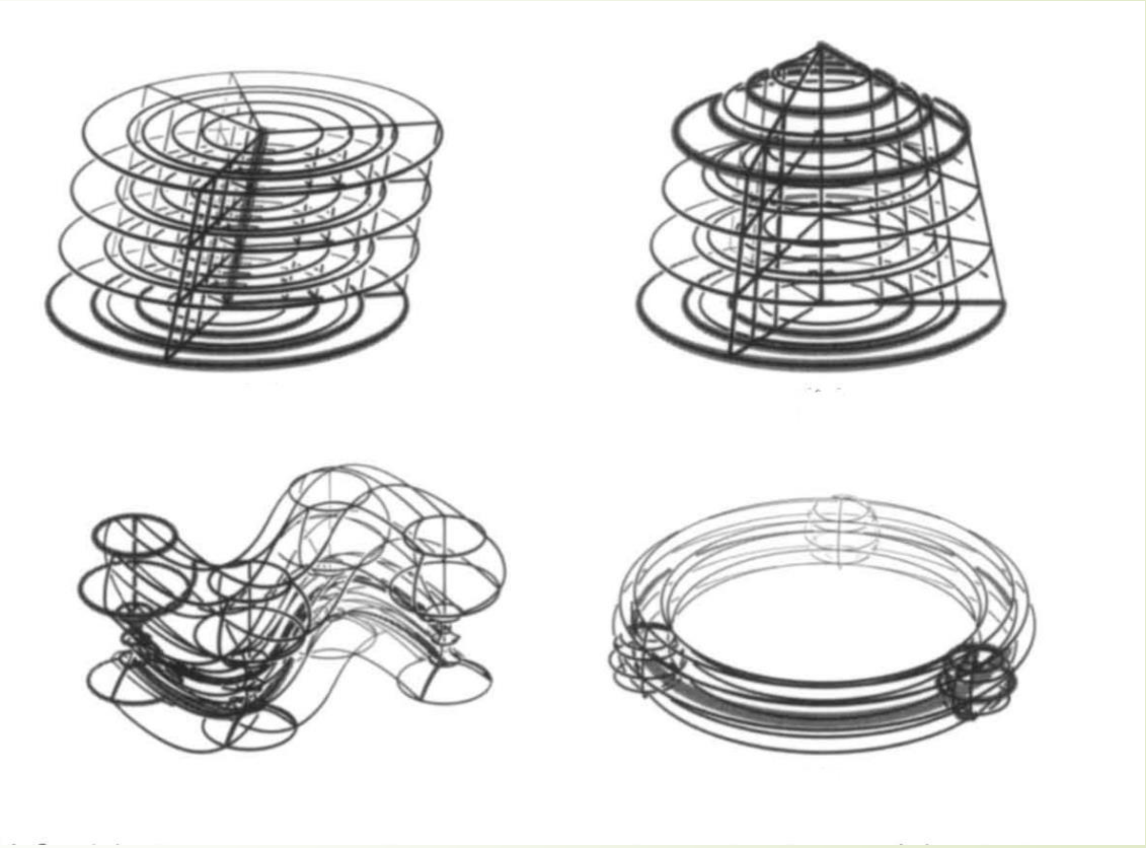
Let T be $T(u, v, w)$: $\mathbb{R}^3 \rightarrow \mathbb{R}^d, \mathrm{d} \geq 3.$

The gradient vector ∇T is called the Jacobian matrix of size $d \times 3$:

$$\nabla T = \left(\frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}\right) = \begin{bmatrix} \sum_i \sum_j \sum_k P_{i,j,k} \grave{B}_i(u) B_j(v) B_k(w) \,, \\ \sum_i \sum_j \sum_k P_{i,j,k} B_i(u) \grave{B}_j(v) B_k(w) \,, \\ \sum_i \sum_j \sum_k P_{i,j,k} B_i(u) B_j(v) \grave{B}_k w) \,, \end{bmatrix}^T .$$

Traditional Constructors

Traditional surface constructors such as **extrusion**, **ruled** surfaces, surfaces of **revolution**, and/or **sweep** surfaces can be extended to construct trivariate functions.

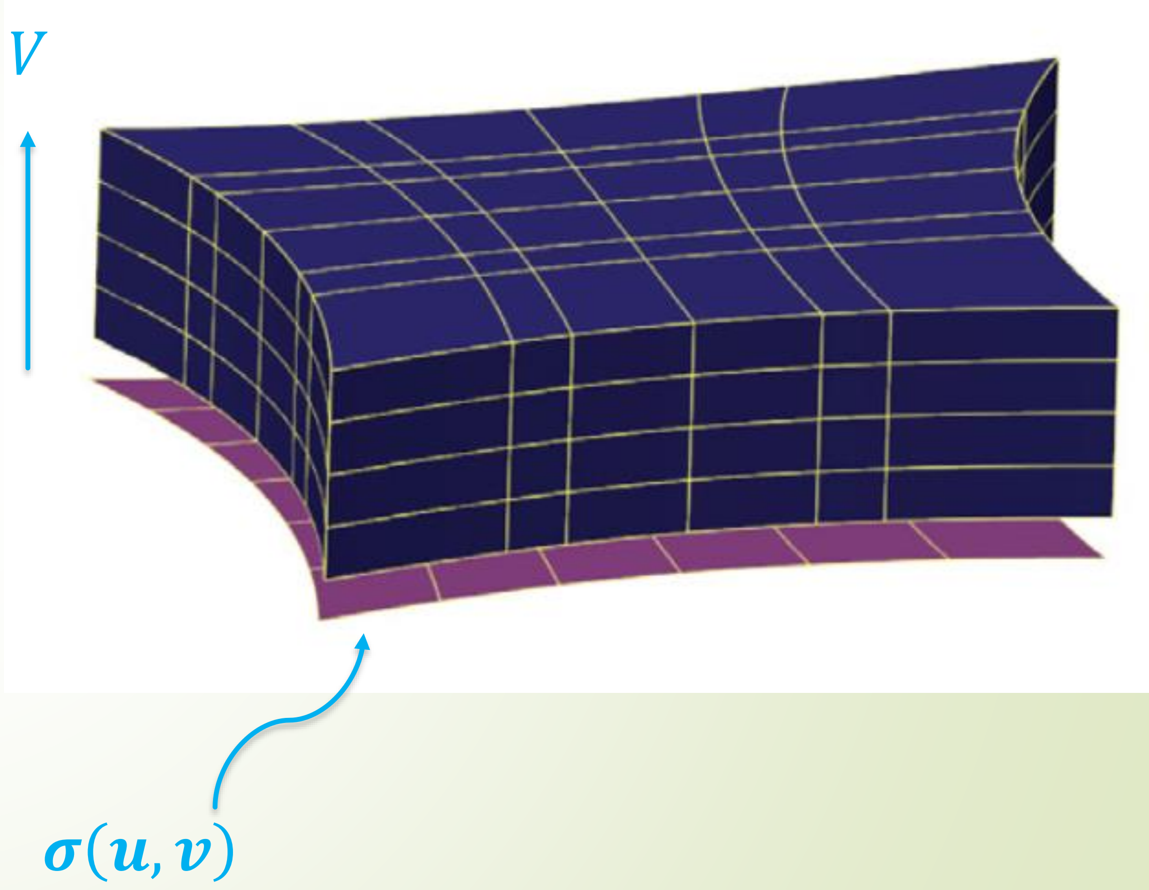


Extruded Volume

An Extruded Volume is a surface crossed with a line. Let $\sigma(u,v)$ and V be a parametric spline surface and a unit vector, respectively. Then

$$\mathrm{T}(u,v,w) = \sigma(u,v) + wV,$$

represent the volume extruded by surface $\sigma(u,v)$ as the surface is moved in direction V.

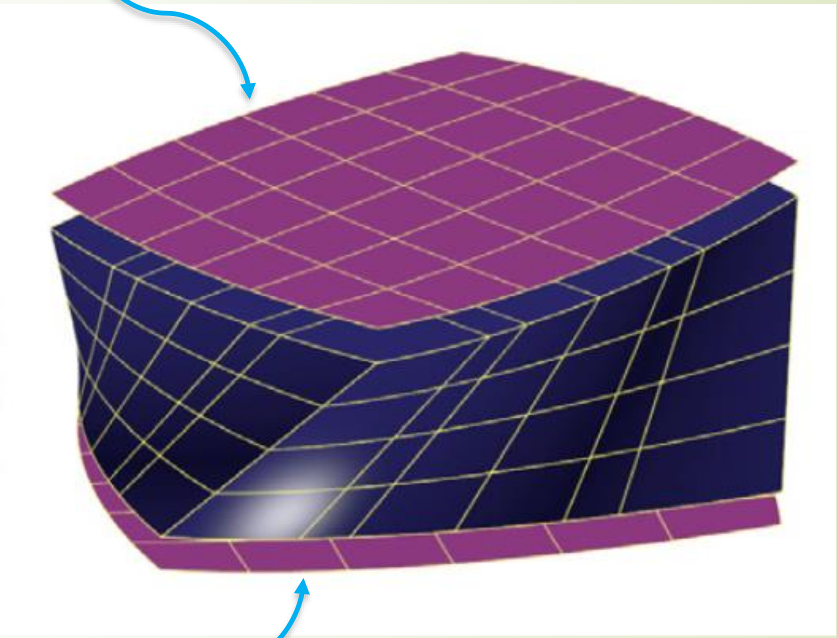


Ruled Volume

Let $\sigma_1(u,v)$ and $\sigma_2(u,v)$ be two parametric spline surfaces **in the same space**, that is, the same order and knot sequences. Then, the trivariate:

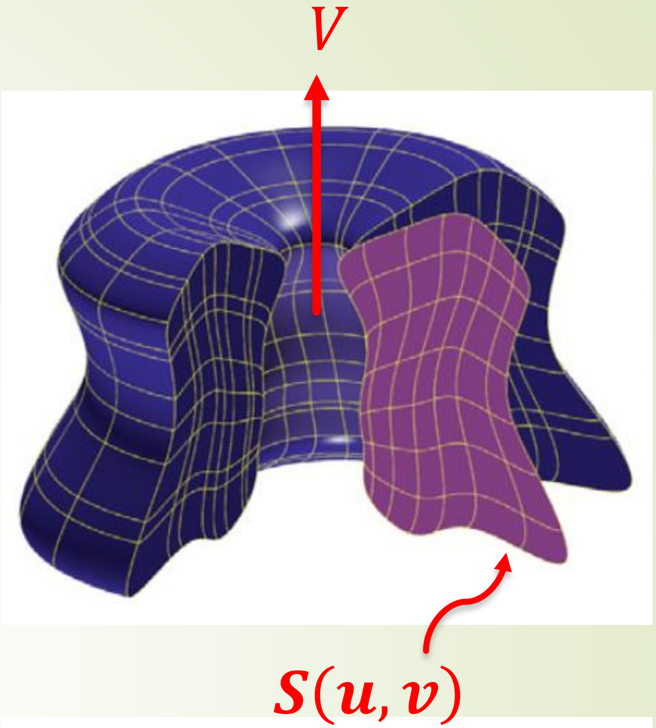
$$\mathrm{T}(u,v,w) = (1 - w)\sigma_1(u,v) + w\sigma_2(u,v),$$

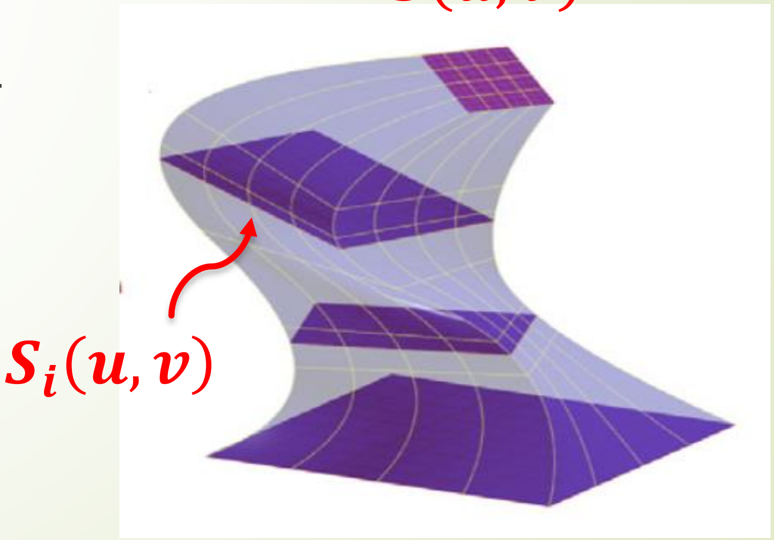
constructs a ruled volume between σ_1 and σ_2. If $\sigma_1(u,v)$ and $\sigma_2(u,v)$ do not share the same function subspace, they both could be elevated into one by raising the degrees of the lower degree surface.



- **Volume of revolution:** Given a surface $S(u, v)$, a trivariate of revolution $T(u, v, w)$ is constructed by rotating S around some axis V.

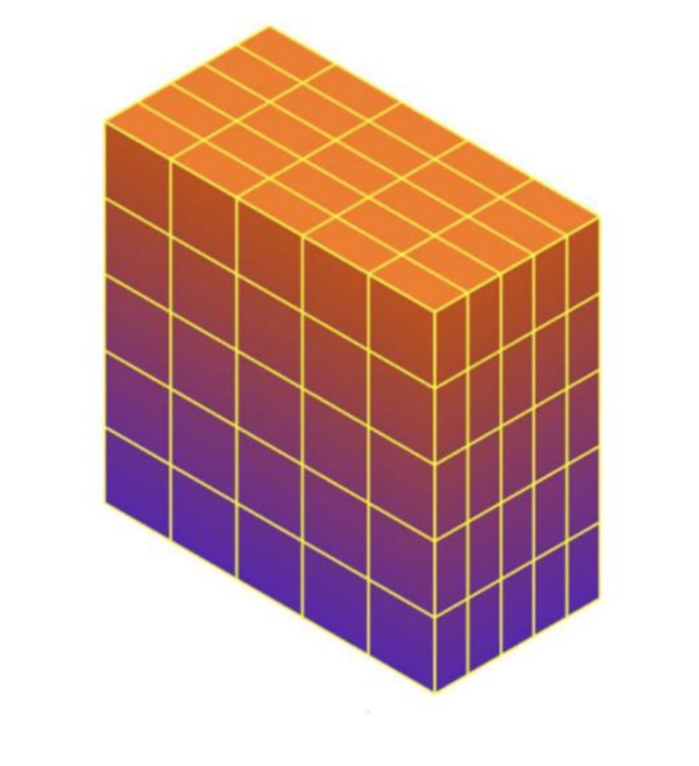
- **Volumetric sweep:** A trivariate T is generated that interpolates or approximates a given ordered list of surfaces, S_i(u, v), at different w_i parameters, w_i ∈ [0, 1].





Volumetric Representation (V-Reps)

The framework is designed to support anticipated AM and IGA needs, which means it should support different queries on V-rep models such as slicing (intersection with a plane), (point) inclusion, geometrical neighborhood information and contacts, local refinements and more.



Based on Elber and Massarwi paper [2].

V-Reps (Cont.)

- A V-model is a complex of several volumetric cells (V-cells).
- The V-cell is a volumetric cell represented by a **trimmed B-spline** trivariate:

  $$T(u,v,w) = \sum_i \sum_j \sum_k P_{i,j,k} B_i(u) B_j(v) B_k(w)$$

  where $P_{i,j,k} \epsilon \mathcal{R}^{\mathfrak{q}}$, $\mathbf{\mathfrak{q}} \geq \mathbf{3}$ where the first three coordinates always represent the geometry but optionally also **additional attributes**, such as a **color** or a **stress tensor field**, for $\mathfrak{q} \geq 3$.
- The B-spline trivariate is our basic building block that defines a volume. However, it is limited to a cuboid topology, and cannot represent general volumetric shapes.
- The trimming of a V-cell is prescribed by a set of trimming (bivariate) B-spline surfaces in the domain of the trivariate. Each such trimming surface is, in turn, possibly trimmed by trimming B-spline curves

V-Reps (Cont.)

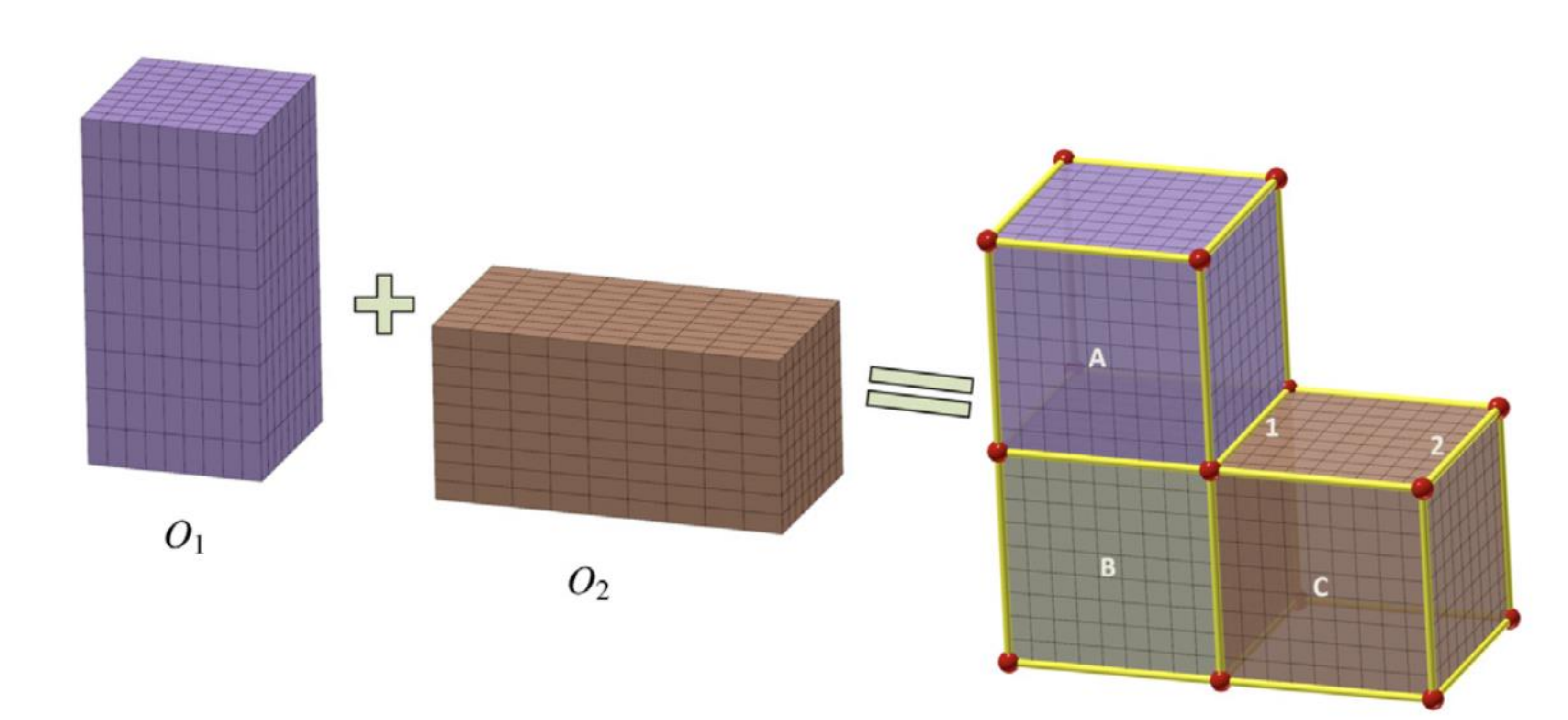
- **Definition 3.2:**

  A V-rep cell (V-cell) is a 3-manifold that is in the intersection of one or more B-spline tensor product trivariates. The sub-domain of the intersection is delineated by trimming surfaces.

- **Definition 3.3:**

  A V-model is a complex of one or more (mutually exclusive) V-cells. Adjacent V-cells possibly share boundary (trimming) surfaces, curves or points.

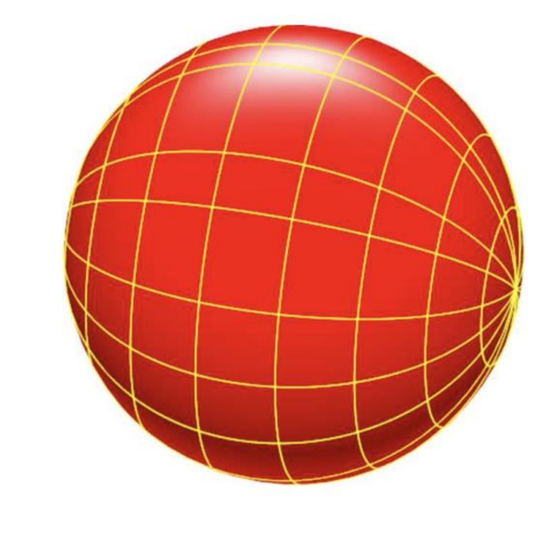
Union Operation between V-Reps



O_1 and O_2 are V-model, and the union between them will resulting three V-cells A, B and C.

What about continuity of attribute fields?

Union Operation (Cont.)



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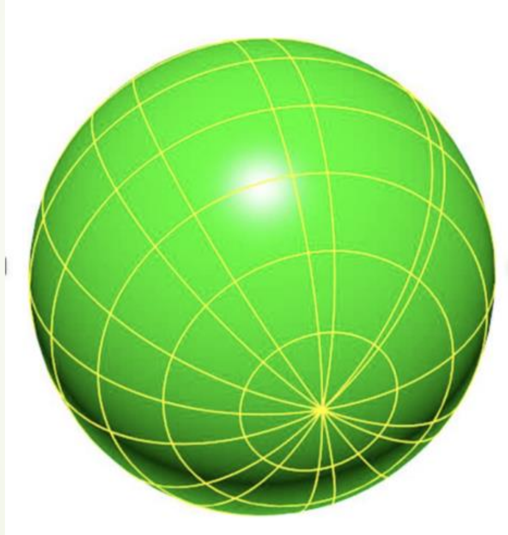

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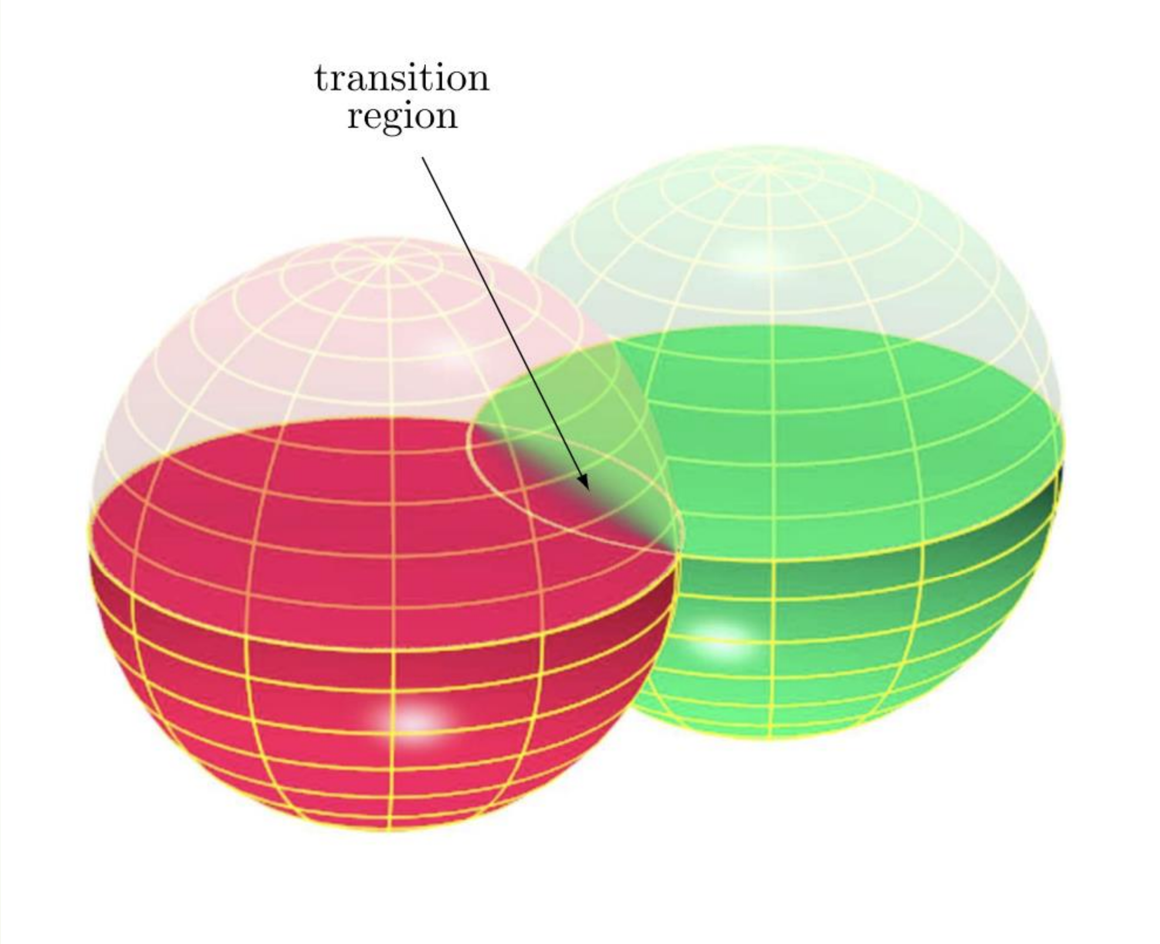
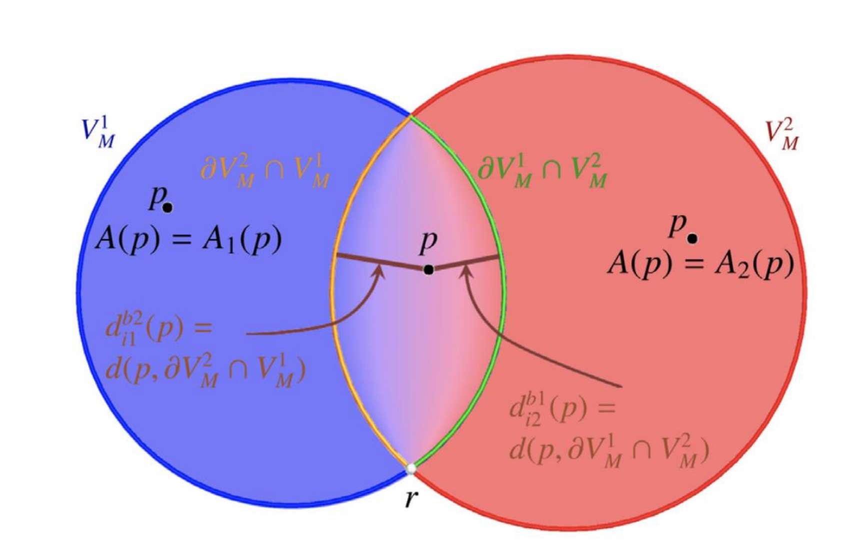


What happen to the attribute fields in this transition region?

Material Heterogeneity

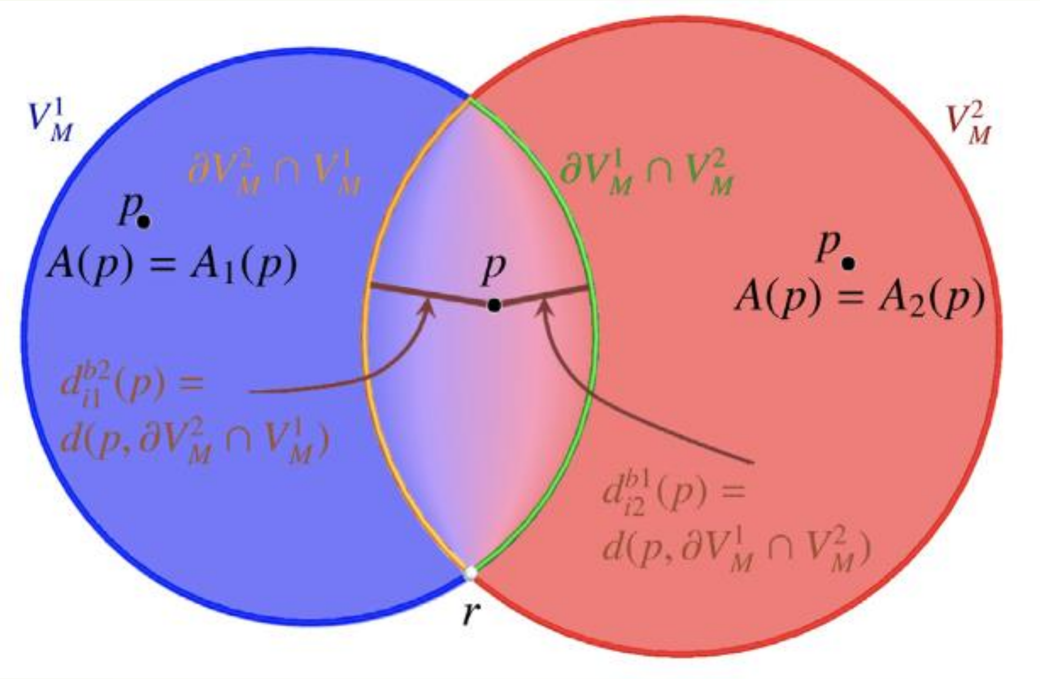
- Let $V_M = V_M^1 \cup V_M^2$, be some V-model that is the result of a Boolean union operation between V-models V_M^1 and V_M^2.
- For simplicity, lets assume each of V_M^1 and V_M^2 contains one V-cell only.
- Let $\boldsymbol{p}$ be a point in V_M, and let $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{O})$ denote the minimal distance from $\boldsymbol{p}$ to object $\boldsymbol{O}$, $\boldsymbol{d}$ is a C^0 continuous function.
- Let $A_1(p)$ and $A_2(p)$ be some C^0 attribute fields of models V_M^1 and V_M^2, respectively, at p, and finally, let $d_{i2}^{b1} = d(p, \partial V_M^1 \cap V_M^2)$ denote the distance from the boundary of V_M^1, ∂V_M^1, that is inside V_M^2 from point p.



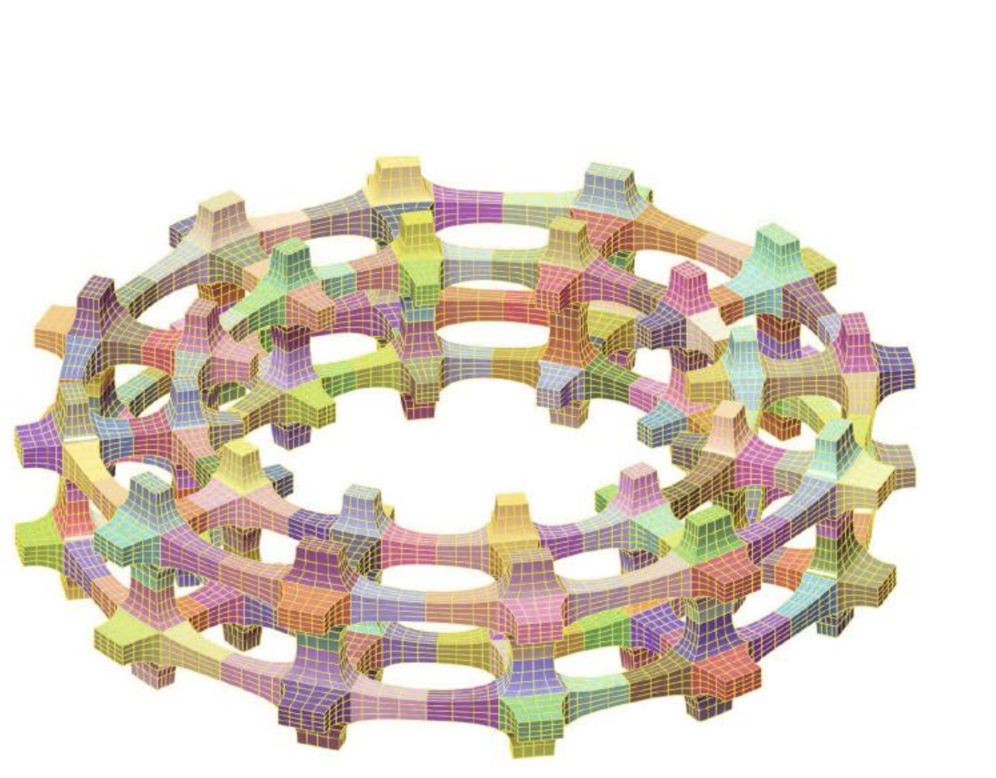
- Then, the attribute value of model V_M , $A(p)$, at p can be calculated as the blend of $A_1(p)$ and $A_2(p)$ as follows:

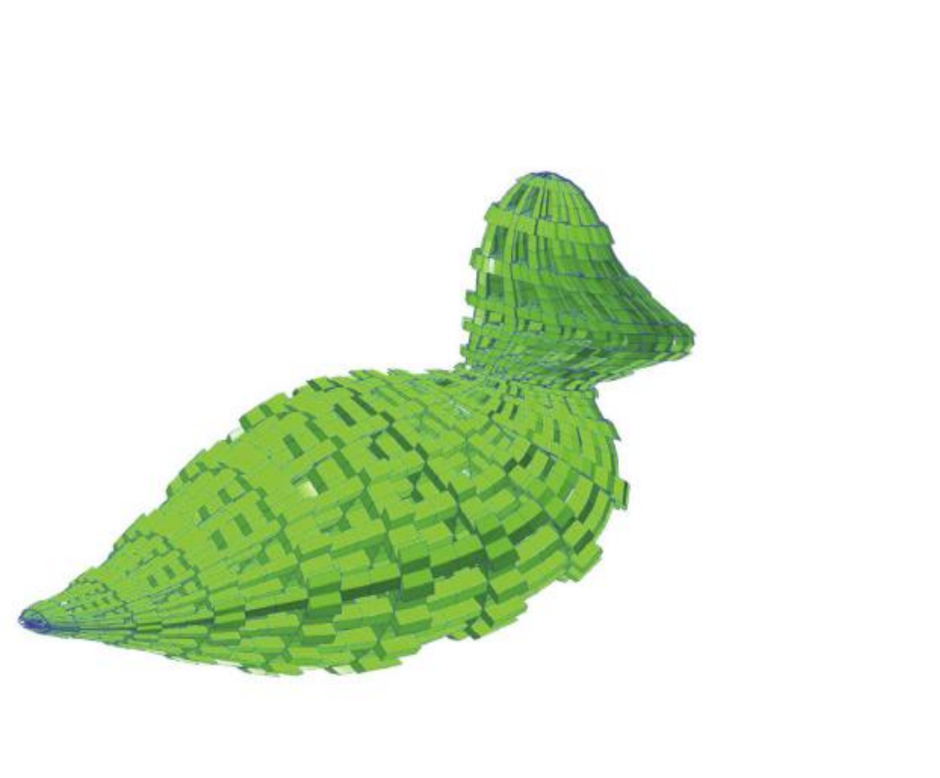
$$A(p) = \begin{cases} \dfrac{d_{i2}^{b1} A_1(p) + d_{i1}^{b2} A_2(p)}{d_{i2}^{b1}(p) + d_{i1}^{b2}(p)}, & p \in V_M^1 \cap V_M^2, \\ A_1(p), & p \in V_M^1 \text{ and } p \notin V_M^2, \\ A_2(p), & p \in V_M^2 \text{ and } p \notin V_M^1, \end{cases}$$

- Points where both $d_{i2}^{b1}(p)$ and $d_{i1}^{b2}(p)$ vanish simultaneously, called singular location.



Construction of Micro-structures via Functional Composition

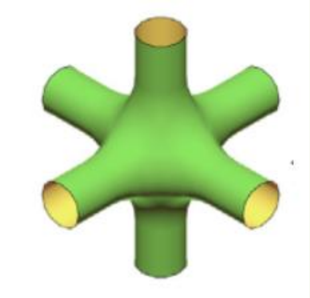




Based on Elber Paper [3].

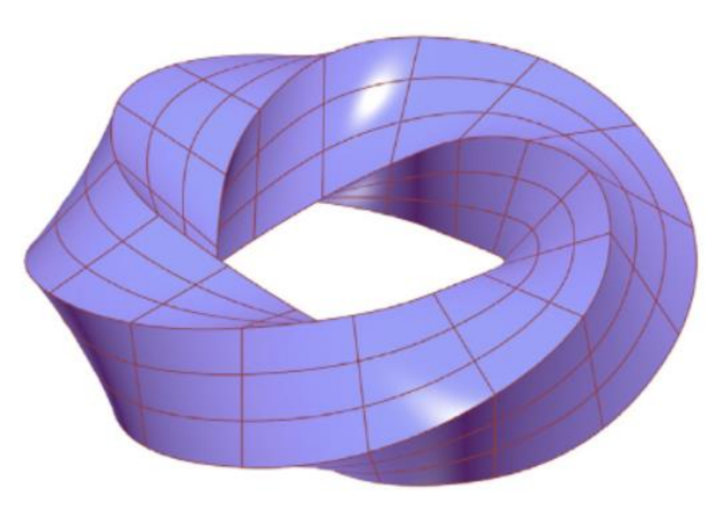
Construction of Micro-structures (Cont.)

- Based on Elber method [3] to build precise construction of micro-structures using functional composition. In that approach, the designs of the macro-shape $\mathcal{T}$ and the micro-structures $\mathcal{M}$ of a porous geometry are decoupled.



$\mathcal{M}$

+



$\mathcal{T}$

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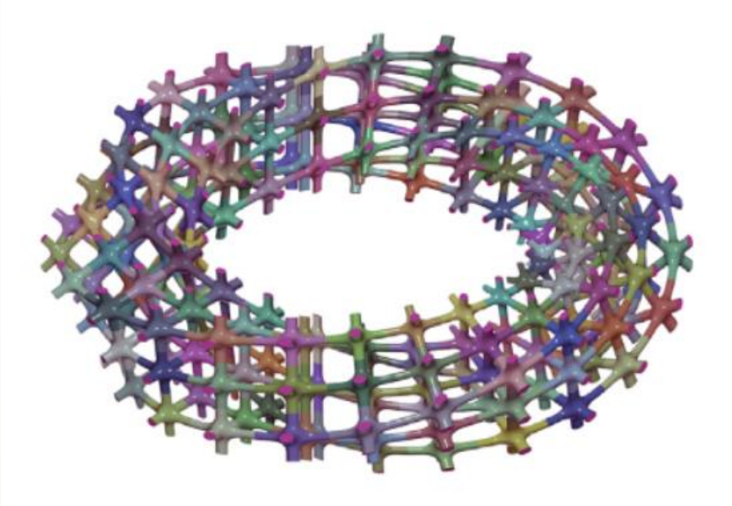
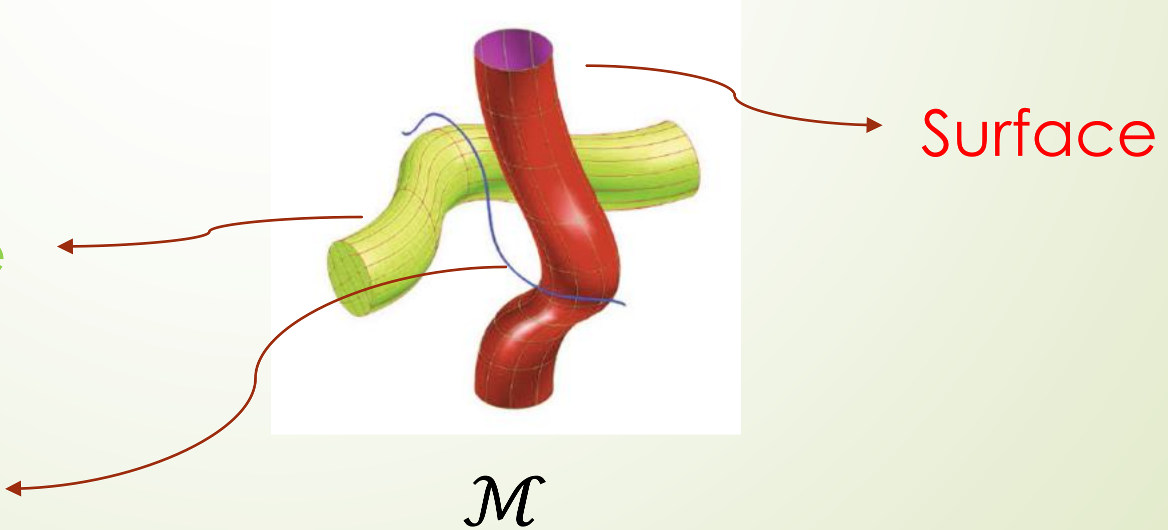


Construction of Micro-structures (Cont.)

- A parametric form of a (**typically periodic**) micro-tile $\boldsymbol{\mathcal{M}}$ is specified as some combination of curves, surfaces, and/or trivariates while the macro-shape $\boldsymbol{\mathcal{T}}$ is also specified as a parametric trivariate function $\mathcal{T}: D \in \mathbb{R}^3 \to \mathbb{R}^3$.



$\mathcal{M}$

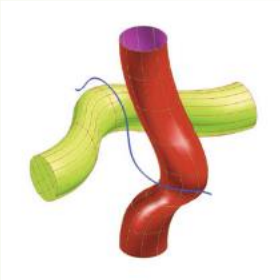
Construction of Micro-structures (Cont.)

Our input set is:

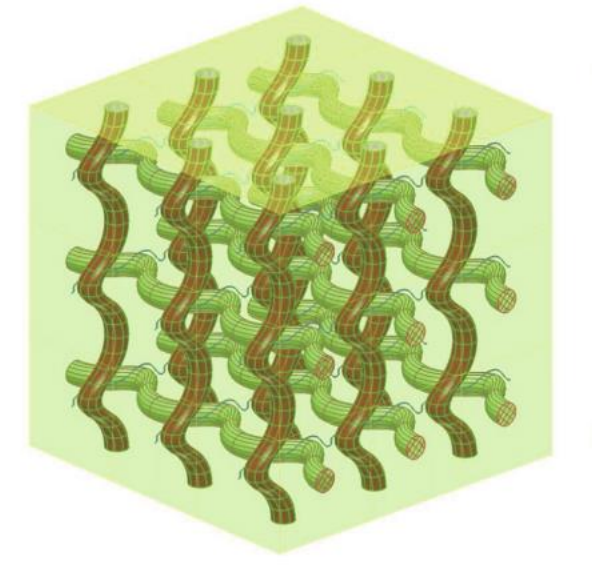
- A micro-tile $\mathcal{M}$, as some combination of parametric curves $C(t) = (c_x(t), c_y(t), c_z(t))$, surfaces $S(u,v) = (s_x(u,v), s_y(u,v), s_z(u,v))$ and trivariates $T(u,v,w) = (t_x(u,v,w), t_y(u,v,w), t_z(u,v,w))$. $\mathcal{M}$ is typically periodic in the sense that the d_{min} faces are C^0- continuous with respect to d_{max}, $d = x, y, z$, and may even be C^k- continuous, $k > 0$.
- A trivariate parametric deformation macro-function $\mathcal{T}(x,y,z): D \in \mathbb{R}^3 \rightarrow \mathbb{R}^3$.
- (n_x, n_y, n_z): the dimensions of enumerations in $\mathcal{T}$, in (x,y,z) of the micro-tile $\mathcal{M}$.

Construction of Micro-structures (Cont.) First Step

- Pave the micro-tile $\mathcal{M}$ in the domain D (of $\mathcal{T}$), (n_x, n_y, n_z) times.



$\mathcal{M}$

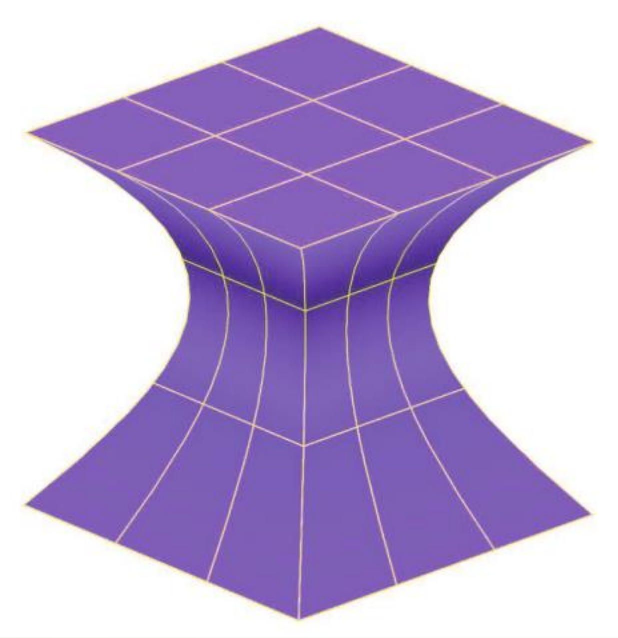


$\mathcal{D}$

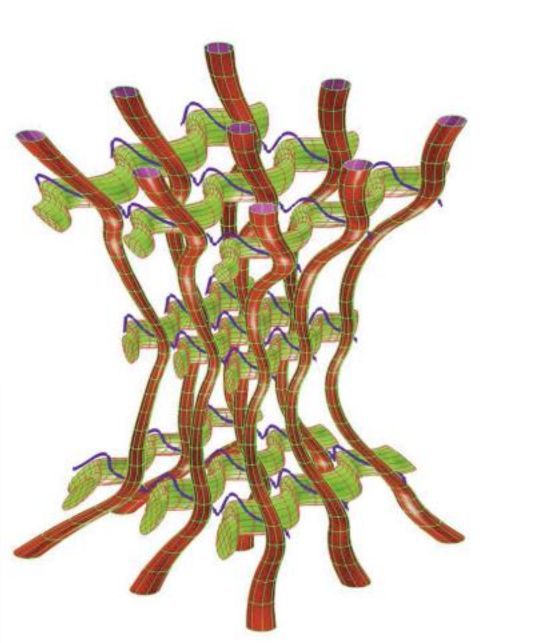
The micro-tile $\mathcal{M}$ paved in $\mathcal{D}$ $(3,3,3)$ times.

Construction of Micro-structures (Cont.) Second Step

- Functionally composing the tiling $\{M_{i,j,k}\}$, into the deformation map $\mathcal{T}$, to obtain the required micro-structure $\mathcal{T}(M_{i,j,k})$.



$\mathcal{T}$



$\mathcal{T}(M)$

Construction of Micro-structures (Cont.) Functional Composing details

- Let $\mathcal{T}$ be a trivariate vector function:

$$\mathcal{T}(u,v,w) = \sum_i \sum_j \sum_k P_{i,j,k} B_i(u) B_j(v) B_k(w)$$

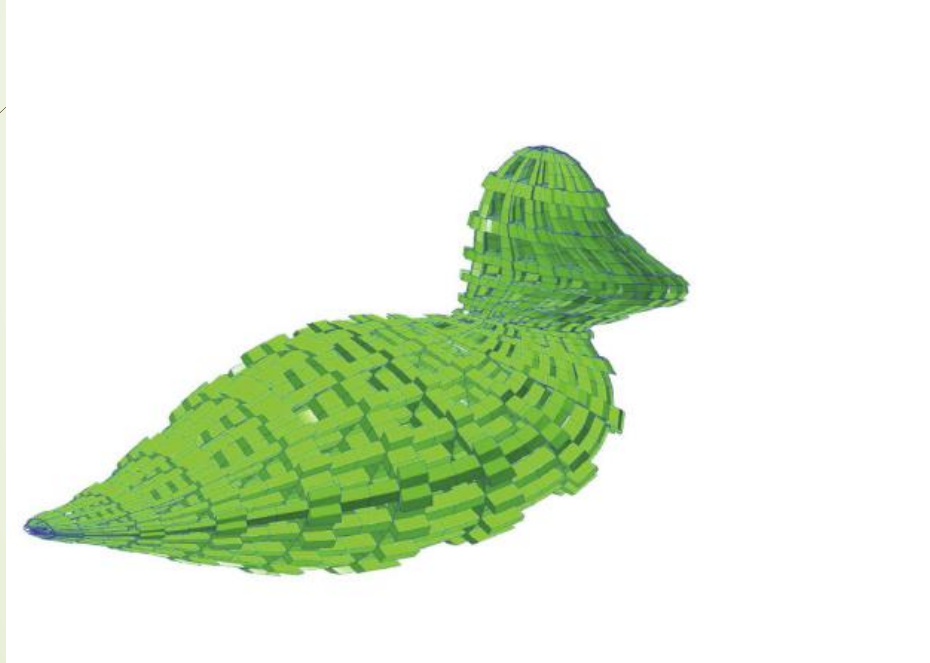
- And the micro-shape trivaiate tile M is:

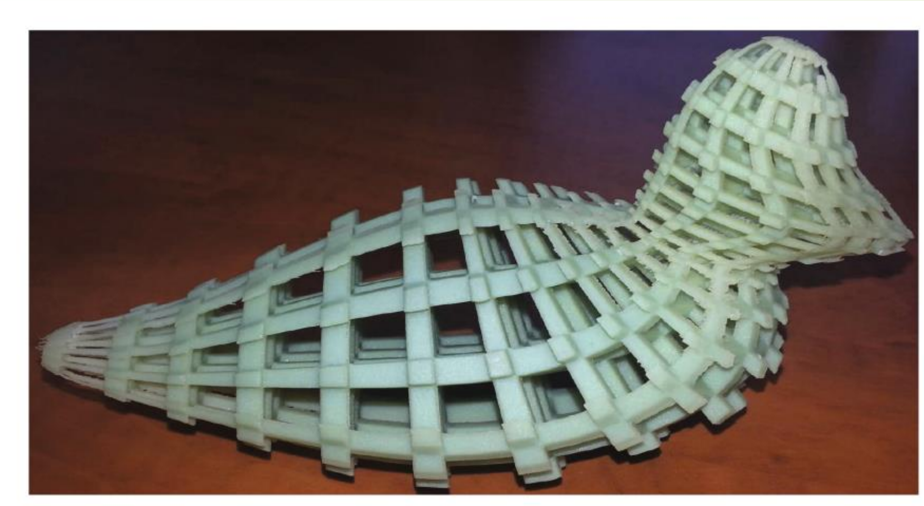
$$\mathrm{M}(u,v,w) = \sum_i \sum_j \sum_k Q_{i,j,k} B_i(u) B_j(v) B_k(w)$$

- Then, the microstructure can be build using function composition:

$$\tilde{T} = \mathcal{T}(M) = \mathcal{T}(m^x(u,v,w), m^y(u,v,w), m^z(u,v,w))$$

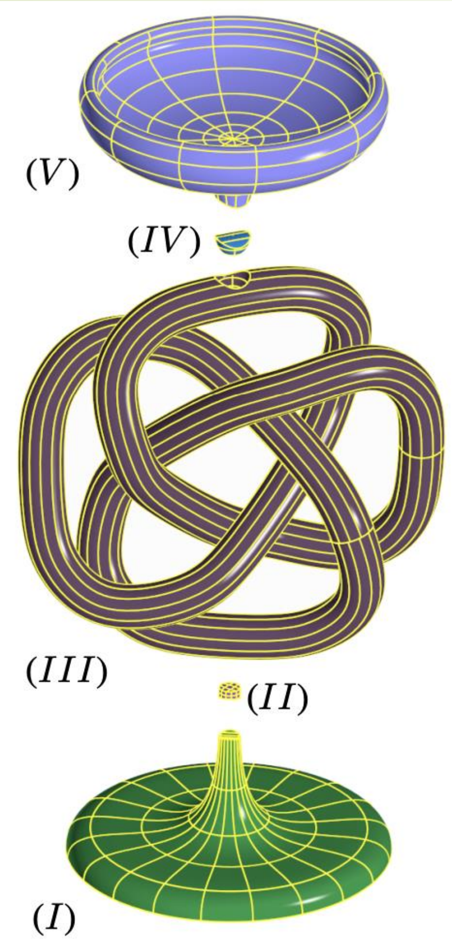
Additive Manufacturing

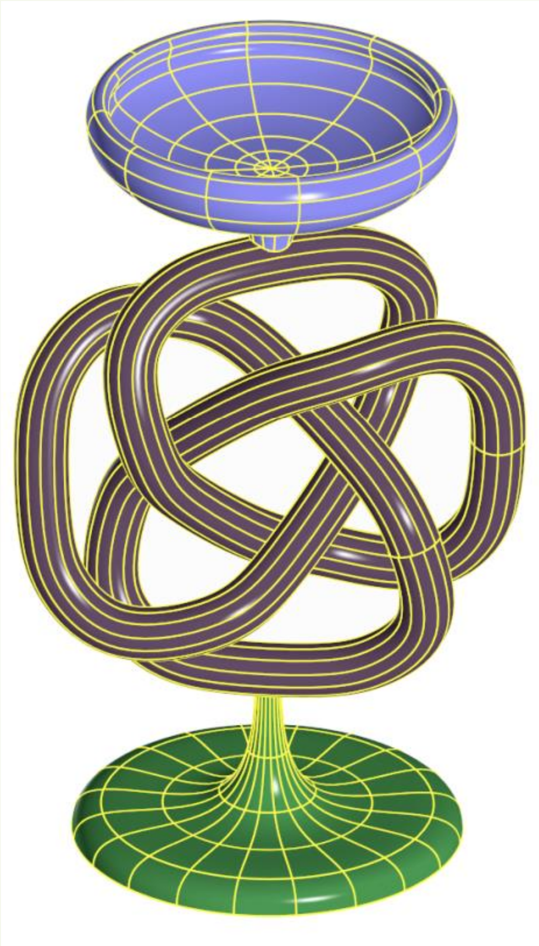




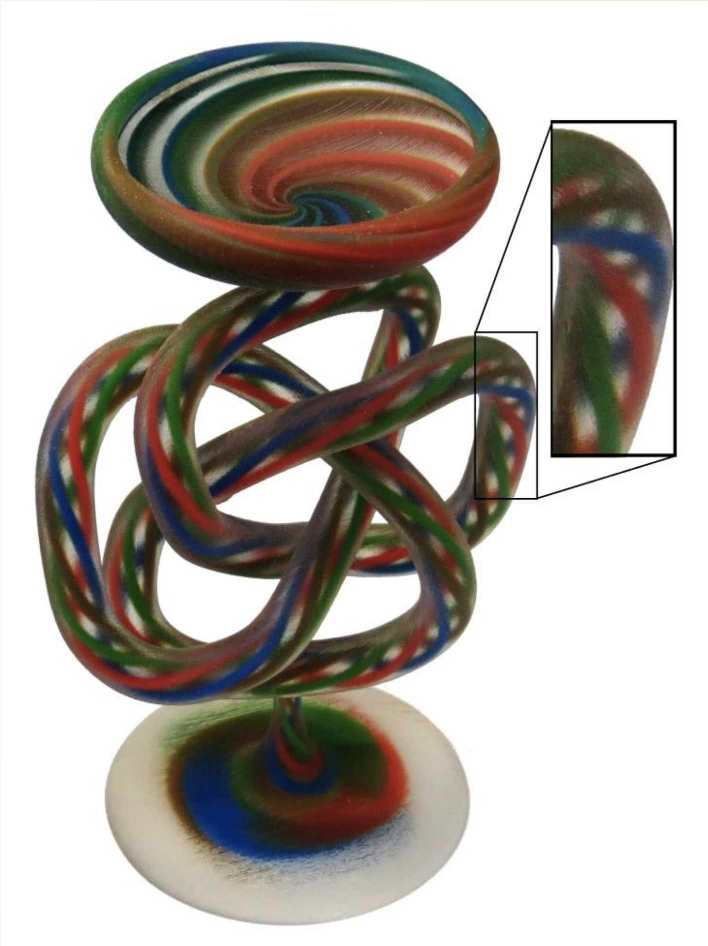
Based on Ezair and Elber paper [4].

Additive Manufacturing (Cont.)





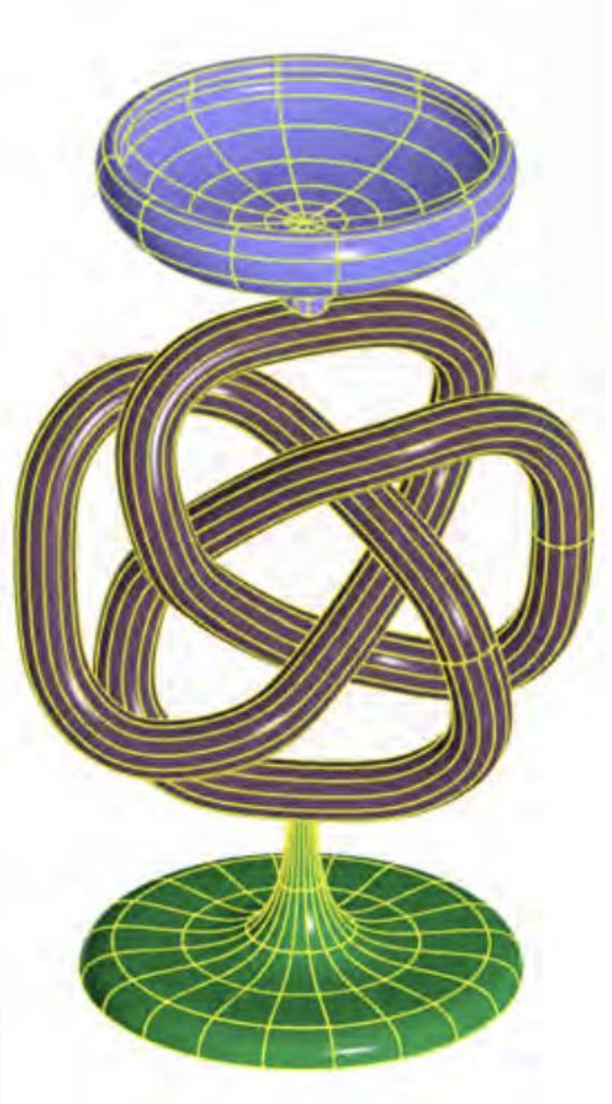
Five V-cells Composed to one V-rep

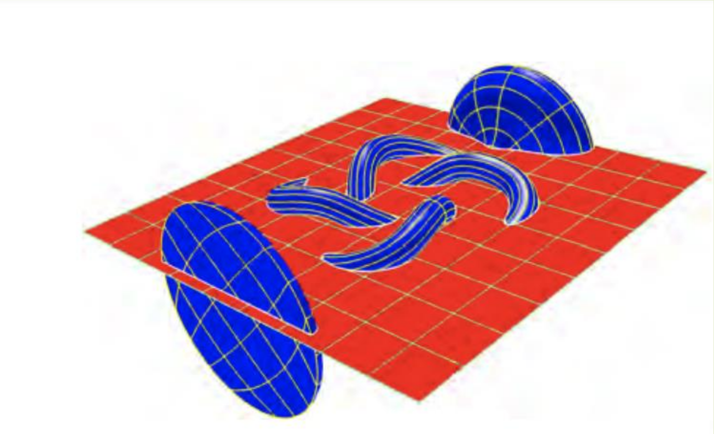


The fabricated graded-materials (FGM) object.

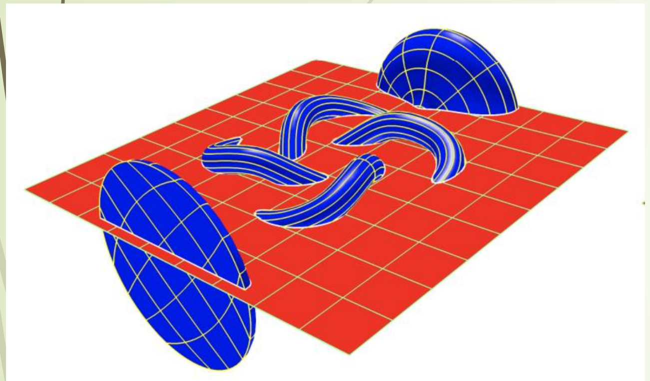
Additive Manufacturing (Cont.)

We apply the standard AM approach of slicing, to convert the above representation of an FGM object to a set of instructions that can be used to manufacture the object using AM.

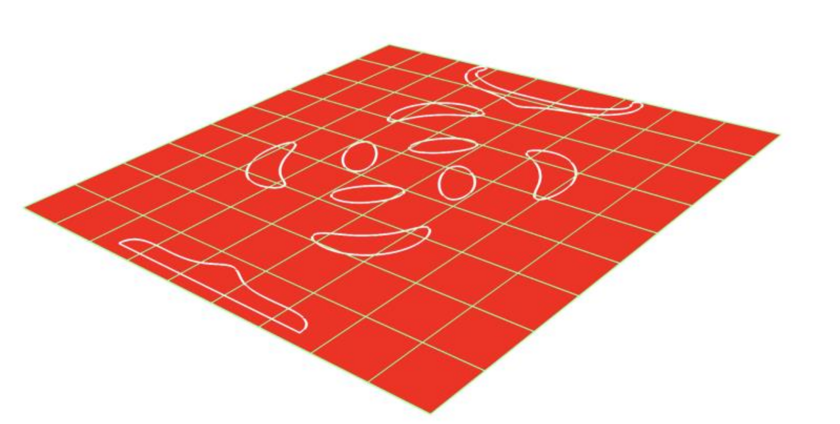


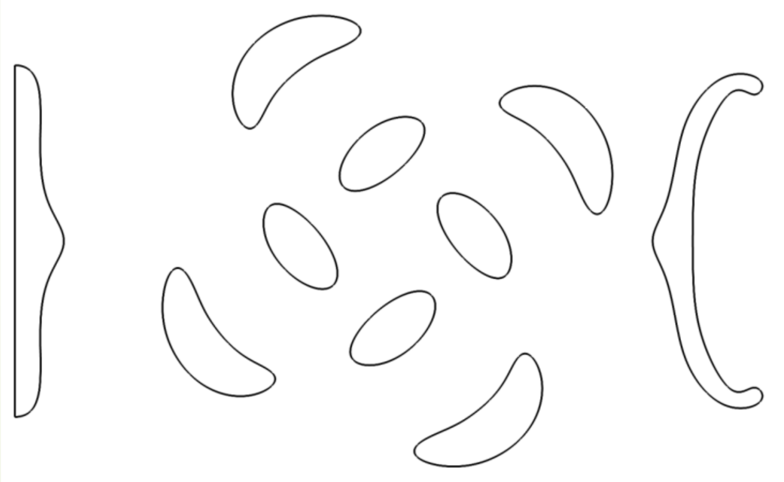


Additive Manufacturing (Cont.)

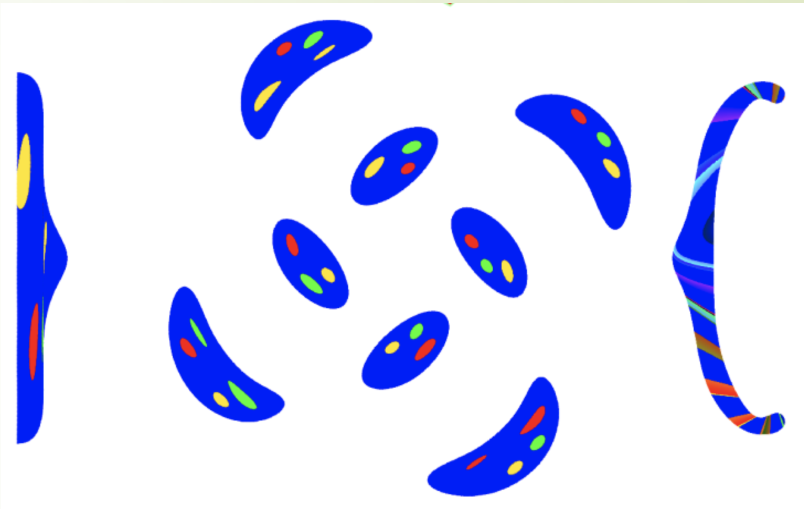


The V-rep model is intersected with a plane.





The planar outline for the slice.



The planar Slice is filled with color coded material information.

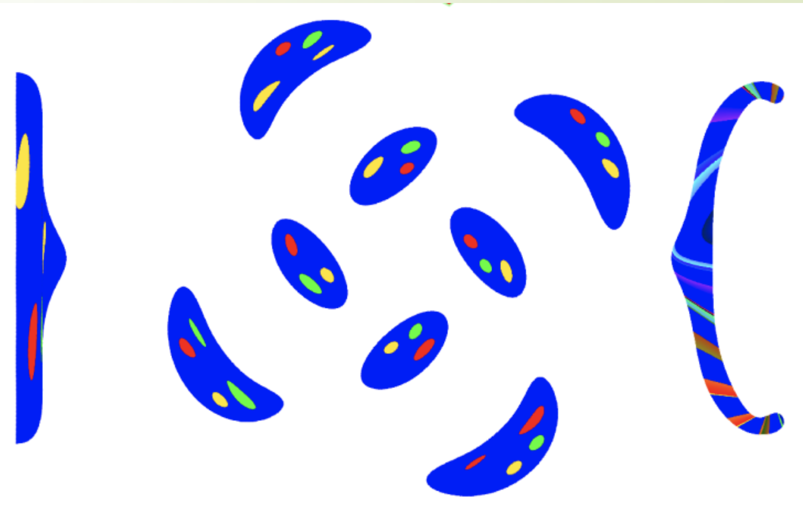
Additive Manufacturing (Cont.)

Printing Pipline:

I. Intersect the V-rep model with a plane (at a given z height), to get a ’slice’.

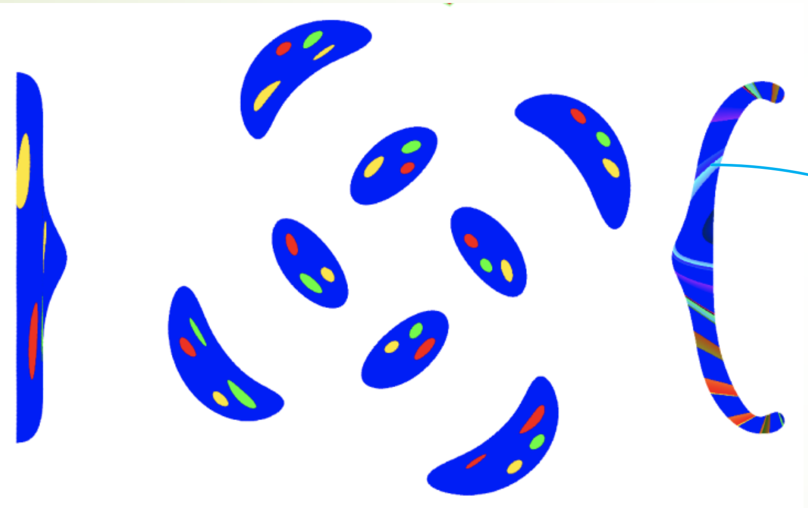
II. This slice is then filled with material information by **evaluating** and specify the material composition **at every point** inside the slice. Then can be used by the printer to manufacture that slice.

How we can evaluate the material composotion on each point inside the ‘slice’ ?



Additive Manufacturing (Cont.) Point Evaluation

- When slicing an FGM object, the only information available for a point is its position in **Euclidean space**.



The location for each point p is $(x, y, \hat{z})$, in the eculidean space. (the z coordinate is fixed for all points in the slice).

But the geometry and material composition functions of the V-rep are over the **Parametric domain**.

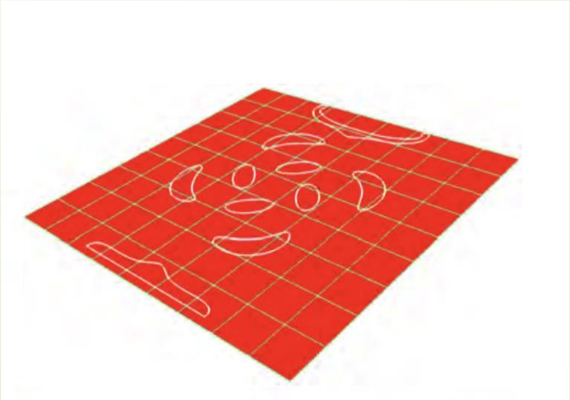
Additive Manufacturing (Cont.)
Point Evaluation

Given a Euclidean point $(x, y, \hat{z})$, consider the following system of three (often non-linear piecewise polynomial) constraints and three unknowns (u, v, w):

$$\begin{cases} x = v_x(u, v, w) \\ y = v_y(u, v, w) \\ y = v_y(u, v, w) \end{cases}, \qquad (1)$$

It's a classical inverse problem, that can be solved **using a subdivision based multivariate solvers.**

We can solve this equation for each point, for billion of times. It is not efficient!



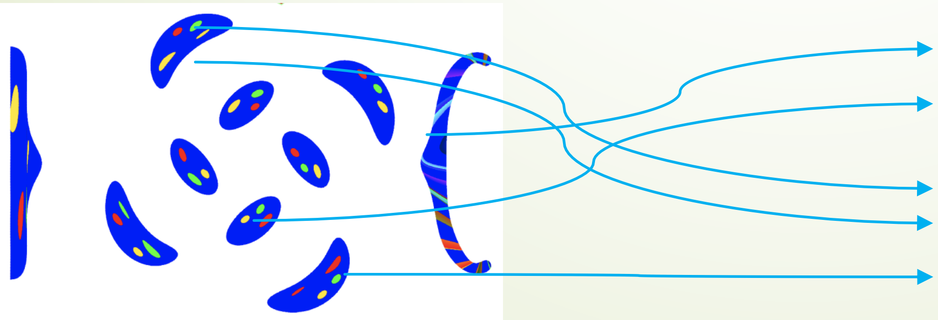
Additive Manufacturing (Cont.)
Point Evaluation

Lemma:

The solution for the system of constraints presented in Equation (1), $(u0, v0, w0)$, **exists and is unique** for every point (x_0, y_0, z_0) inside a self intersection free and regular volume $V(u, v, w)$.

Proof:

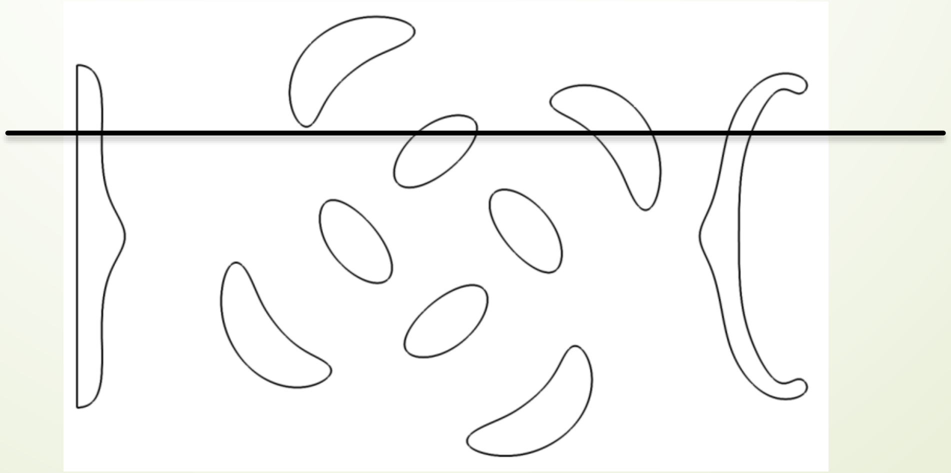
The uniqueness of the solutions to Equation (1) follows directly from our assumption of $V(u, v, w)$ being regular and without (global) self intersections.



For each point (x_0, y_0, z_0) **inside the outline** of the objcet have a un unique solution.

Additive Manufacturing (Cont.)
Point Evaluation

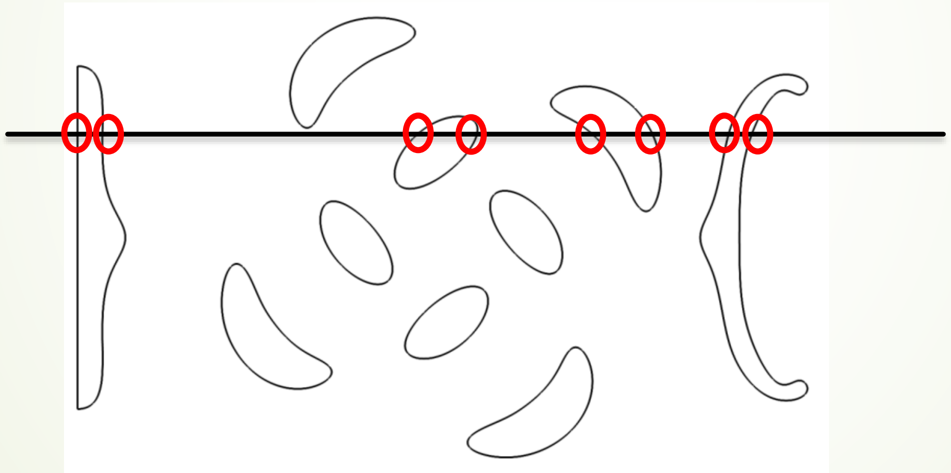
- To efficiently classify points as inside or outside the model, we use the outline of the current slice.
- Given the outline, we aim to sample and generate material composition information **only for the pixels** that are found **inside the outline** of the slice.
- We use a simple rasterization strategy of sampling along straight **lines aligned to the y axis**.



Additive Manufacturing (Cont.)
Point Evaluation

Sampling along lines allows us to use two optimizations:

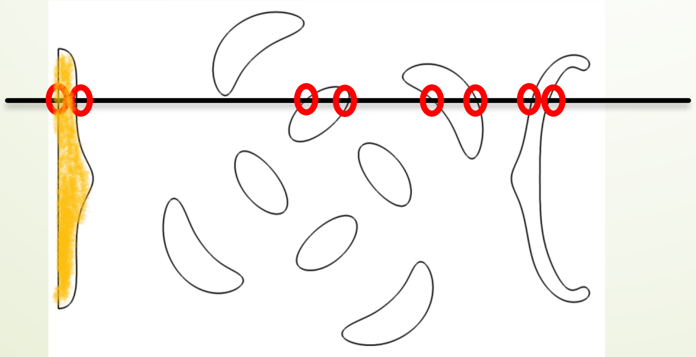
I. To determine if we are inside/outside the outline, we first find the intersections between the outline and our sampling line.



Using this optimization to determine if a sample point is **inside or outside** the current slice **saves us from unnecessarily** attempting to solve Equation (1) for points outside the current slice.

Additive Manufacturing (Cont.)
Point Evaluation

II. From the previous lemma we already know that a unique solution exists if the Euclidean point is inside.
   - This fact allows us to optimize our use of the multivariate solver to solve equation (1).
   - Once we use the full solver to get the solution for a point, the solution for any other close-by point can be attempted by employing a numeric tracing (i.e. a Newton–Raphson iteration method), using the previous solution for any close-by point as an initial guess.



Refrences:

1. “Geometric Modeling with Splines, An Introduction" by Cohen, Riesenfeld, Elber: Chapter 21.
2. Elber, G., Massarwi, F., 2016. A B-spline based framework for volumetric object modeling. Computer Aided Des. 78, 36–47 .
3. Elber G. Precise construction of micro-structures and porous geometry via functional composition. In: Proceedings of the 9th international conference on mathematical methods for curves and surfaces; 2016. p. 108–25.
4. Ezair, B., Elber, G., 2017. Fabricating functionally graded material objects using trimmed trivariate volumetric representations, in: Proceedings of SMI’2017 Fabrication and Sculpting Event (FASE), Berkeley, CA, USA.